1994 ALGEBRA PRELIMINARY EXAMINATION

Group Theory

1. Show that there is no simple group of order 20.

2. (a) Let G be a group. For each $x \in G$, let $\bar{x} = \{gxg^{-1} : g \in G\}$. Prove that $|\bar{x}| = [G : C(x)]$ where $C(x) = \{g \in G : gxg^{-1} = x\}$.
(b) Prove that center of a nontrivial finite p-group is nontrivial (where p is a prime).

3. Suppose N is a normal subgroup of a group G. Prove that G/N is abelian if and only if $G' \subseteq N$, where G' denotes the derived subgroup of G.

4. (a) Prove that $\mathbf{Z}/m\mathbf{Z} \times \mathbf{Z}/n\mathbf{Z} \cong \mathbf{Z}/mn\mathbf{Z}$ if m and n are relatively prime positive integers.
(b) Give a list of abelian groups, each of order 540, such that each abelian group of order 540 is isomorphic to exactly one member of the list.

5. Prove that every cyclic group is isomorphic to $\mathbf{Z}/n\mathbf{Z}$ for some nonnegative integer n.

6. (a) Define *indecomposable* group.
(b) Show that the additive group of rational $\mathbf{Q}$ is indecomposable.

7. Prove or disprove: There exists a nontrivial homomorphism $f : S_5 \to \mathbf{Z}/5\mathbf{Z}$, where S_5 is the symmetric group on 5 letters.

8. Let $\mathcal{C}$ be a concrete category and let X be a nonempty set.
   (a) Let F be an object of $\mathcal{C}$ and let $i : X \to F$ be a set map. Define what it means to say that (F, i) is *free* on X.
   (b) If (F, i) and (F', i') are both free on X, prove that $F \cong F'$.
   (c) Does there exist a pair (F, i) that is free on X in the category having abelian groups as objects and homomorphisms as morphisms? Explain.

Ring Theory

In this section, R is a ring with 1, and all modules are unitary left R-modules.

1. Verify the following
   (a) For every module M, there exists a free module F and an epimorphism $\phi : F \to M$.
   (b) If M is a simple module and $\theta : M \to N$ is a homomorphism, then either $\theta = 0$ or $\theta(M) \cong M$.
   (c) If R is semisimple, conclude that every module is isomorphic to a direct sum of minimal left ideals of R.

2. (a) Show that an injective module is divisible.
   (b) If R is a PID, show that every divisible module is injective.
   (c) Show that every free module is projective.

3. Let J denote the Jacobson radical of R.
   (a) If $J = 0$, show that for each $x \in R$ there exists a maximal left ideal of R that excludes x. Conclude that R embeds into a direct sum of simple left R-modules.
   (b) If M is a finitely generated module and $JM = M$, show that $M = 0$. (Hint: induct on the minimal number of generators for M.)

Field Theory

1. Prove:
   (a) Every finite field extension is algebraic.
   (b) There exists an algebraic field extension that is not finite.

2. Suppose F is a finite field of p^n elements for some prime p and positive integer n. Show that F contains a subfield with p^m elements if and only if $m|n$.

3. Find the Galois group of each polynomial $f(x)$ over the given field K and determine whether $f(x) = 0$ is solvable by radicals over K. By finding the Galois group, we mean finding a "familiar"group to which the Galois group is isomorphic.
   (a) $f(x) = x^4 - 6$, $K = \mathbf{Q}$
   (b) $f(x) = x^4 - 6$, $K = \mathbf{Q}(i)$
   (c) $f(x) = x^5 - 4x^4 + 2x^3 + 2x^2 + x + 6$, $K = \mathbf{Q}$
   (d) $f(x) = x^{27} - x$, $K = \mathbf{Z}/3\mathbf{Z}$
   (e) $f(x) = x^5 + 5x^3 - 20x + 5$, $K = \mathbf{Q}$

4. Suppose $K \subseteq F$ is a field extension and $\overline{K}$ is an algebraic closure of K (which does not necessarily contain F). Prove that there exists a K-monomorphism $\sigma : F \to \overline{K}$ if and only if F is algebraic over K.

5. Let F be a splitting field over $\mathbf{Q}$ of the polynomial $f(x) = x^5 - 3 \in \mathbf{Q}[x]$. Use the Fundamental Theorem of Galois Theory to show that the Galois group $\text{Aut}_{\mathbf{Q}}F$ of $f(x)$ over $\mathbf{Q}$ contains a subgroup of order 4 that is not normal. Conclude that $\text{Aut}_{\mathbf{Q}}F$ is not abelian.

6. Recall that a finite field extension is Galois if and only if it is both normal and separable.
   (a) Give an example of a finite field extension which is separable but not Galois.
   (b) Give an example of a finite field extension which is normal but not Galois.