# Linear Algebra Preliminary Exam, 2011
# August 20, 2011, 9:00 a.m.-12:00 p.m., Parker 250
# Professor T.Y. Tam

Name:

**For full credit, show all steps in details**

**Choose 5 out of 7**

1. (a) Define unitary and normal matrices.

   (b) Prove Schur's triangularization theorem by induction: For $A \in M_n(\mathbb{C})$, there is a unitary matrix $U \in M_n$ such that $U^*AU$ is upper triangular.

   (c) State the real analog of Schur's triangular theorem for $A \in M_n(\mathbb{R})$.

   (d) Use Schur's triangularization to prove the spectral theorem for normal matrices, i.e., for each normal $A \in M_n$, there is a unitary matrix $U \in M_n$ such that $U^*AU$ is diagonal.

2. (a) State Jordan canonical form theorem for $A \in M_n$.

   (b) Use the Jordan canonical form to show that each $A \in M_n$ is similar to its transpose $A^T$.

   (c) What are the possible Jordan forms of a matrix $A \in M_n$ with characteristic polynomial $p_A(t) = (t+3)^4(t-5)^2$?

   (d) If the minimal polynomial of $A \in M_n$ is $q_A(t) = \prod_{i=1}^{m}(t-\lambda_i)^{r_i}$, where $\lambda_1, \ldots, \lambda_m$ are distinct eigenvalues of $A$, then what is $r_i$ in terms of the Jordan blocks of $A$ corresponding to the eigenvalues $\lambda_i$? Explain.

3. (a) Show that each $A \in M_n$ can be written uniquely as $A = H + iK$ where $H$ and $K$ are Hermitian matrices.

   (b) Show that $A \in M_n$ is Hermitian if and only if $x^* A x$ is real for all $x \in \mathbb{C}^n$.

   (c) Show that all the eigenvalues of a Hermitian $A \in M_n$ are real.

   (d) Use interlacing inequalities and induction to show Schur's theorem: Let $A \in M_n$ be Hermitian. The diagonal $d := (a_{11}, \ldots, a_{nn})^T$ of $A$ is majorized by the eigenvalues $\lambda \in \mathbb{R}^n$ of $A$.

4. (a) Show that all vector norms on a finite dimensional vector space $V$ are equivalent, i.e., there are $m, M > 0$ such that $m\|x\|_\alpha \leq \|x\|' \leq M\|x\|$ for all $x \in V$, where $\|\cdot\|$ and $\|\cdot\|'$ are any two vector norms on $V$.

   (b) What is the difference between a matrix norm $|||\cdot|||$ on $M_n$ and a vector norm on $M_n$? Give an example that is a vector norm on $M_n$ but not a matrix norm.

   (c) Show that if $|||\cdot|||$ is a matrix norm on $M_n$, then $\rho(A) \leq |||A|||$ where $\rho(A)$ is the spectral radius of $A$.

   (d) Is the spectral radius a norm? Explain or give a counterexample.

5. (a) State and prove Gersgorin theorem on $A \in M_n$. What happens when we apply the theorem on $D^{-1}AD$ where $D = \operatorname{diag}(p_1, \ldots, p_n)$ and $p_1, \ldots, p_n$ are nonzero?

   (b) Suppose that $A \in M_n$ is a real matrix whose $n$ Gersgorin discs are all mutually disjoint. Show that all the eigenvalues of $A$ are real.

   (c) Show that if $A$ is strictly diagonally dominant, then $A \in M_n$ is nonsingular.

   (d) Prove that if $A \in M_n$, then $\rho(A) \le \min\{\max_i \sum_{j=1}^n |a_{ij}|, \max_j \sum_{i=1}^n |a_{ij}|\}$.

6. (a) Prove SVD for $A \in M_{m,n}$: $A = V\Sigma W^*$, $V \in M_m$, $W \in M_n$ unitary, $\Sigma$ "diagonal".

   (b) Show that if $\Sigma = \begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix}$ where $S = \operatorname{diag}(\sigma_1, \ldots, \sigma_r)$ and $r$ is the rank of $A \in M_{m,n}$, then $A = V_1 S W_1^*$ where $V = [V_1\ V_2]$, $V_1 \in M_{m,r}$ and $W = [W_1\ W_2]$, $W_1 \in M_{n,r}$.

   (c) Deduce polar decomposition ($A = PU$, $P \in M_m$ positive semidefinite and $U \in M_{m,n}$ has orthonormal rows) from SVD.

7. (a) State Perron theorem (6 statements) on positive matrices $A \in \mathbb{R}_{n\times n}$. When $A \in \mathbb{R}_{n\times n}$ is irreducible nonnegative, which statements remain true? Give counterexamples to those among the six which not true anymore.

   (b) Suppose that $A \in M_n(\mathbb{R})$ is irreducible nonnegative and that $B \geq 0$ commute with $A$. If $x$ is the Perron vector of $A$, prove that $Bx = \rho(B)x$, where $\rho(B)$ is the spectral radius of $B$.

   (c) What can we say about the $p$ eigenvalues of an irreducible nonnegative and nonsingular $A \in M_p(\mathbb{R})$ where $p$ is a prime number? Why?