Linear Algebra Preliminary Examination, Spring 2000 (May 13)
Professor T.Y. Tam

Student's Name:

Choose THREE.

1. (a) Define majorization between $x, y \in \mathbb{R}^n$. Give two equivalent statements for $x \prec y$ other than Schur-Horn's result. (5 points)

   (b) State Schur-Horn's result. (4 points)

   (c) By Schur's result, prove Ky Fan's maximum principle for an $n \times n$ Hermitian A:

$$\sum_{j=1}^{k} \lambda_j^{\downarrow}(A) = \max_{\{x_1,\ldots,x_k\} \text{ o.n. in } \mathbb{C}^n} \sum_{j=1}^{k} x_j^* A x_j, \quad k = 1, \ldots, n, \qquad \text{(6 points)}$$

   (d) Deduce from (c) that if A and B are $n \times n$ Hermitian matrices, $\lambda^{\downarrow}(A+B) \prec \lambda^{\downarrow}(A) + \lambda^{\downarrow}(B)$ where $\lambda^{\downarrow}(A)$ denotes the eigenvalue element of A whose entries are arranged in descending order. (5 points)

   (e) Deduce from (d) the corresponding result for the singular values of $A+B$, A and B if A and B are $n \times n$ complex matrices and prove it by using Wieldant's matrix $\begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix}$. (5 points)

2. (a) State Marriage Theorem (Hall's Theorem) on compatible matching. (5 points)

   (b) From Marriage Theorem derive König-Frobenius Theorem: Let $A = (a_{ij})$ be an $n \times n$ matrix. If $\sigma \in S_n$, $(a_{1\sigma(1)}, \ldots, a_{n\sigma(n)})$ is called a diagonal of A. Every diagonal of A contains a zero element if and only if A has a $k \times \ell$ submatrix with all entries zero for some k, ℓ such that $k + \ell > n$. (7 points)

   (c) Show that the set of $n \times n$ doubly stochastic matrices, Ω_n, is a convex set. (4 points)

   (d) Prove, by using König-Frobenius Theorem, that the extreme points of Ω are the permutation matrices. (9 points)

3. (a) Define symmetric gauge functions $\Phi : \mathbb{R}^n \to \mathbb{R}_+$. (4 points)

   (b) Are symmetric gauge functions continuous? Why? (2 points)

   (c) Show that a norm $\|\cdot\| : \mathbb{C}^n \to \mathbb{R}_+$ is absolute, i.e., $\|x\| = \||x|\|$ for all $x \in \mathbb{C}^n$ if and only if it is montone, i.e., $\|x\| \le \|y\|$ whenever $|x| \le |y|$. Then deduce that symmetric gauge functions is monotone. (8 points)

   (d) Prove that if $x, y \in \mathbb{R}_+^n$, then $x \prec_w y$ if and only if $\Phi(x) \le \Phi(y)$ for every symmetric gauge function Φ. (6 points)

(e) Using (d) to show that $\Phi_\infty(x) \le \Phi(x) \le \Phi_1(x)$ for any symmetric gauge function Φ where $\Phi_\infty(x) = \max_{j=1,\dots,n} |x_j|$ and $\Phi_1(x) = \sum_{j=1}^n |x_j|$ (Hint: You may assume that $x \in \mathbb{R}_+^n$). (5 points)

4. (a) Define $|A|$ and polar decomposition of A via singular value decomposition. (4 points)

   (b) Prove Fan-Hoffman's theorem: If $A \in \mathbb{C}_{n\times n}$, then $\lambda_j^\downarrow(\mathrm{Re}\,(A)) \le s_j(A)$, $j = 1,\dots,n$, where $\lambda_j^\downarrow(A)$ denotes the jth largest eigenvalue of $\mathrm{Re}\,A = \frac{1}{2}(A + A^*)$ and s_j denotes the jth largest singular value of A. (7 points)

   (c) Prove $|\lambda(\mathrm{Re}\,A)| \prec_w s(A)$ (Hint: Ky Fan's k-norm). (5 points)

   (d) Let X, Y be Hermitan matrices. Suppose that their eigenvalues can be arranged so that $\lambda_j(X) \le \lambda_j(Y)$ for all j. Show that there exists a unitary U such that $X \le U^*YU$, i.e., $U^*YU - X$ is p.s.d. (3 points)

   (e) Use (b) and (d) to show that for each A there exists a unitary U such that $\mathrm{Re}\,A \le U^*|A|U$. (2 points)

   (f) Then use (e) to prove Thompson's Theorem: If $A, B \in \mathbb{C}_{n\times n}$, then $|A + B| \le U^*|A|U + V^*|B|V$ for some unitary U and V. (4 points)

5. Let Φ be a norm on $\mathbb{C}^n$.

   (a) Define the dual norm Φ' of Φ. (3 points)

   (b) Show that Φ' is a norm. (5 points)

   (c) What is the dual norm of Φ_p, the ℓ_p norm, $1 \le p \le \infty$? (2 points)

   (d) Show that $|(x, y)| \le \min\{\Phi'(x)\Phi(y), \Phi(x)\Phi'(y)\}$ for all $x, y \in \mathbb{C}^n$. (5 points)

   (e) Then show that $\Phi''(x) \le \Phi(x)$ for all $x \in \mathbb{C}^n$. (4 points)

   (f) Show that if Φ and Ψ are two norms such that $\Phi(x) \le c\Psi(x)$ for all $x \in \mathbb{C}^n$ and for some $c > 0$, then $\Phi'(x) \ge c^{-1}\Psi'(x)$ for all $x \in \mathbb{C}^n$. (6 points)

6. (a) Define Schatten p-norm $\|\cdot\|_p$ and Ky Fan k-norm $\|\cdot\|_{(k)}$. Are they unitary invariant (u.i.) norms? (4 points)

   (b) Given a symmetric gauge function Φ on $\mathbb{R}^n$, define $\|\cdot\|_\Phi : \mathbb{C}_{n\times n} \to \mathbb{R}_+$ by $\|A\|_\Phi = \Phi(s(A))$. Show that $\|\cdot\|$ is a unitarily invariant norm such that $\|\mathrm{diag}\,(1, 0, \dots, 0)\| = 1$. (7 points)

   (c) Given a unitarily invariant norm $\|\cdot\| : \mathbb{C}_{n\times n} \to \mathbb{R}$ such that $\|\mathrm{diag}\,(1, 0, \dots 0)\| = 1$, define $\Phi : \mathbb{C}^n \to \mathbb{R}_+$ by $\Phi_{\|\cdot\|}(x) = \|\mathrm{diag}\,x\|$. Show that $\Phi_{\|\cdot\|}$ is a symmetric gauge function. (6 points)

   Remark: Part (b) and (c) constitute von Neumann's Theorem on the characterization of u.i. norms.

(d) Show that (i) $\Phi_{\|\cdot\|_\Phi} = \Phi$ if Φ is a symmetric gauge function and (ii) $\|\cdot\|_{\Phi_{\|\cdot\|}} = \|\cdot\|$ if $\|\cdot\|$ is a unitarily invariant norm. (8 points)