Topology Preliminary Exam, August 2009

Do seven of the eleven problems, including at least one of numbers 9, 10, 11.

1. Let X and Y be compact topological spaces. Prove that $X \times Y$ is compact.

2. Prove that if X is a topological space, Y is a connected subset of X, and $Y \subseteq Z \subseteq \overline{Y}$, then Z is also connected.

3. Prove that a topological space is compact if and only if every nonempty collection of closed sets having the finite intersection property has nonempty intersection.

4. Prove that every separable metric space is second countable.

5. Prove that every paracompact Hausdorff space is also regular.

6. Let X be an uncountable set, with $c \in X$. Define a topology on X by letting all points other than c be isolated, and if $c \in U \subseteq X$, then U is open if and only if $X \setminus U$ is finite. Prove that X is not metrizable.

7. Let X and Y be metric spaces. Prove that a function $f : X \to Y$ is continuous if and only whenever $\langle x_n : n \in \omega \rangle$ is a sequence converging in X to x, $\langle f(x_n) : n \in \omega \rangle$ converges in Y to $f(x)$.

8. Let X_n be metric spaces, $n \in \omega$. Prove that $\Pi_{n \in \omega} X_n$ is metrizable.

9. Let $p : (E, e_0) \to (B, b_0)$ be a covering map, and let $f : [0,1] \to B$ be a path with initial point b_0. Prove that there is a path $\tilde{f} : [0,1] \to E$ with initial point e_0 such that $p \circ \tilde{f} = f$.

10. Let $p : (E, e_0) \to (B, b_0)$ be a covering map, let f and g be paths in B with initial point b_0, and let $\tilde{f}$ and $\tilde{g}$ be the lifts of f and g (respectively) to E with initial point e_0. Prove that if f and g are path-homotopic in B then $\tilde{f}$ and $\tilde{g}$ have the same final point.

11. Let A be a deformation retract of a space X. Prove that the inclusion map $i : A \to X$ induces an isomorphism of the fundamental groups of A and X.