Topology Preliminary Exam.

August 11, 2007.

Do 8 of the following problems. The proofs are to be constructed from the definitions and standard elementary properties of topological spaces. If in doubt whether or not something may be assumed, please ask.

Show your work and give your reasoning; full credit may not be given if the reasoning is incorrect or incomplete.

Assume that all spaces are Hausdorff.

If you cannot do the problem as stated but are able to do the problem with an added topological condition then do so and you may receive partial credit.

1.) Prove: if X is compact and Y is compact then $X \times Y$ is compact.

2.) Prove: if X is a compact space then it is not the union of countably many non-where dense sets.

3.)Prove that a paracompact space is regular.

4.)Prove: a metric space is regular and first countable.

5.)Prove: a separable metric space has a countable basis of open sets.

6.)Prove: the continuous image of a connected space is connected.

7.)Prove: if $f : X \to Y$ is a continuous 1 to 1 onto function and X is compact, then f^{-1} is continuous. Show that the statement is not true if the condition of compactness is removed.

8.)Let $\{A_i\}_{i=1}^{\infty}$ be a sequence of connected sets such that for each positive integer j, $A_j \cap A_{j+1} \neq \emptyset$ then $\cup_{i=1}^{\infty} A_i$ is connected.

9.) Prove the path lifting theorem: Let X be a covering space for Y with covering map p and let $p(x_0) = y_0$; let $h : I \to Y$ be a path with initial point y_0. Then there is a unique lifting $\widetilde{h}$ of h to X so that $\widetilde{h}(0) = x_0$.

10.) Prove: $\pi_1([0, 1] \times X) = \pi_1(X)$.

11.) State the van Kampen theorem and use it to calculate the $\pi_1(X)$ for X where X is the union of the two-sphere (e.g. $\{(x, y, z) \in E^3 | x^2 + y^2 + z^2 = 1\}$) and the unit circle so that they have a single point in common.

12.) Suppose that for each i X_i is a separable space and $f_i : X_{i+1} \to X_i$ is an onto bonding map then prove that the inverse limit space $X = \varprojlim \{X_i, f_i\}_{i=1}^{\infty}$ is separable. Show that X need not be separable if the bonding maps are not onto. (Note for the sake of this problem the "empty space" is assumed to be separable.)

13.) Show either:

a.) A separable metric space is Lindelöf;

or b.) A metric Lindelöf space is separable.