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Preliminary Doctoral Examination in Partial Differential Equations

Name: ______________________________

Work Problems 1 and 2 and at least one of Problems 3–5. No books or notes are allowed.

Throughout, $\mathbb{R}^N$ is Euclidean N-space, with $N \in \mathbb{N}$, $N \geq 2$, and Ω is a bounded domain in $\mathbb{R}^N$ with boundary $\partial\Omega$. Whenever it exists, $\hat{n}$ denotes the unit outward normal vector field on $\partial\Omega$.

Problem 1

(a) Suppose that $u, v \in L^1_{loc}(\Omega)$, $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{Z}_+^N$. What do we mean when we say that $D^\alpha u = v$ in the weak sense? (Notation: $D^\alpha = \partial_1^{\alpha_1} \cdots \partial_N^{\alpha_N}$)

(b) Given $p \in [1, \infty]$ and $m \in \mathbb{Z}_+$, give the definition of the Sobolev space $W^{m,p}(\Omega)$ and its norm, denoted by $\| \cdot \|_{m,p}$.

(c) What do we mean when we say that Ω is a Lipschitz domain?

(d) Suppose Ω is a Lipschitz domain and let $p \in [1, \infty)$, $u \in W^{1,p}(\Omega)$. Then $u|_{\partial\Omega}$ is defined in the sense of traces and belongs to $L^p(\partial\Omega)$. Explain what this means and why it is true.

(e) Under the assumptions of (d), the boundary condition $u = 0$ on $\partial\Omega$ may be interpreted in the sense of traces. There is another interpretation, equivalent under the assumptions of (d), that applies to arbitrary domains, without any regularity condition. Explain!

(f) Assuming again that Ω is Lipschitz, let $p \in [1, \infty)$, $u \in W^{2,p}(\Omega)$. Then $\frac{\partial u}{\partial \hat{n}}$ is defined in the sense of traces and belongs to $L^p(\partial\Omega)$. Explain why.

(g) Assuming that Ω is Lipschitz, let $p, p' \in (1, \infty)$, $1/p + 1/p' = 1$, $u \in W^{1,p}(\Omega)$, $\bar{v} \in W^{1,p'}(\Omega)^N$, $\phi \in W^{2,p}(\Omega)$, $\psi \in W^{1,p'}(\Omega)$. Integrate by parts: $\int_\Omega (\nabla u) \cdot \bar{v} = \ldots$, $\int_\Omega (\Delta\phi)\psi = \ldots$

(h) Assuming that Ω is Lipschitz, let $p, q \in [1, \infty)$. Under what condition on p and q is $W^{1,p}(\Omega)$ contained in $L^q(\Omega)$? Under what conditions is the embedding continuous or even compact?

(i) Suppose Ω is such that $W^{1,p}(\Omega)$ embeds compactly into $L^p(\Omega)$. Let $p \in [1, \infty)$ and assume that $| \cdot |$ is a continuous seminorm on $W^{1,p}(\Omega)$ with $|1_\Omega| \neq 0$, where 1_Ω is the constant function with value 1 on Ω. For $u \in W^{1,p}(\Omega)$, define $|u|_{1,p} := (\sum_{j=1}^N \|\partial_j u\|_p^p)^{1/p}$ and $\|u\| := (|u|^p + |u|_{1,p}^p)^{1/p}$. Prove that $\| \cdot \|$ is a norm on $W^{1,p}(\Omega)$, equivalent to $\| \cdot \|_{1,p}$. Conclude that, restricted to the hyperplane $\{u \in W^{1,p}(\Omega) \mid \int_\Omega u = 0\}$ or any other closed complement of $\mathbb{R}1_\Omega$ in $W^{1,p}(\Omega)$, $| \cdot |_{1,p}$ and $\| \cdot \|_{1,p}$ are equivalent norms.

Hint. The only difficulty is to show that there exists a constant c such that $\|u\|_{1,p} \leq c\|u\|$ for all $u \in W^{1,p}(\Omega)$. Assuming the contrary, construct a sequence (u_n) in $W^{1,p}(\Omega)$ with $\|u_n\|_{1,p} = 1$ for all n and $\|u_n\| \to 0$ as $n \to \infty$. Using the compactness of the embedding of $W^{1,p}(\Omega)$ into $L^p(\Omega)$, argue that a subsequence of (u_n) converges with respect to $\| \cdot \|_{1,p}$, necessarily with limit 0.

Problem 2

(a) Suppose that Ω is Lipschitz. Given $\lambda \in \mathbb{R}$ and a function $f \in L^2(\Omega)$, consider the Neumann problem

$$(\mathrm{P}_\lambda) \qquad -\Delta u + \lambda u = f \text{ in } \Omega, \quad \frac{\partial u}{\partial \hat{n}} = 0 \text{ on } \partial\Omega.$$

Recall that by a *strong solution* of (P_λ), we mean a function $u \in H^2(\Omega)$, satisfying $-\Delta u + \lambda u = f$ in the sense of weak derivatives and $\frac{\partial u}{\partial \hat{n}} = 0$ in the sense of traces. Show that a function $u \in H^2(\Omega)$ is a strong solution of (P_λ) if and only if

$$(*) \qquad \int_\Omega \big\{ (\nabla u) \cdot (\nabla v) + \lambda u v \big\} = \int_\Omega f v \quad \text{for all } v \in H^1(\Omega).$$

(b) Recall that a function $u \in H^1(\Omega)$ that satisfies $(*)$ is called a *weak solution* of (P_λ). More generally, given $\lambda \in \mathbb{R}$ and a functional $\phi \in H^1(\Omega)^*$, the weak problem associated with (P_λ) is to find $u \in H^1(\Omega)$ such that

$$(\mathrm{Q}_\lambda) \qquad \int_\Omega \big\{ (\nabla u) \cdot (\nabla v) + \lambda u v \big\} = \phi(v) \quad \text{for all } v \in H^1(\Omega).$$

Prove that if $\lambda > 0$, then Problem (Q_λ) is *well posed*, that is, for every $\phi \in H^1(\Omega)^*$ there is a unique solution $u \in H^1(\Omega)$, and the mapping $\phi \mapsto u$ is continuous with respect to the norm topologies of $H^1(\Omega)^*$ and $H^1(\Omega)$.

(c) For the case $\lambda = 0$, prove the following: Given $\phi \in H^1(\Omega)^*$, Problem (Q_0) has a solution in $H^1(\Omega)$ if and only if $\phi(1_\Omega) = 0$, where 1_Ω is the function with constant value 1 on Ω. Further, if $\phi(1_\Omega) = 0$ and u_0 is one solution of (Q_0), then the set of all solutions is given by $u_0 + \mathbb{R}1_\Omega$. In particular, if $\phi(1_\Omega) = 0$, then (Q_0) has a unique solution u_0 in the orthogonal complement of $\mathbb{R}1_\Omega$ in $H^1(\Omega)$, and the mapping $\phi \mapsto u_0$ is continuous with respect to the norm topologies of $H^1(\Omega)^*$ and $H^1(\Omega)$.

(d) For arbitrary $\lambda \in \mathbb{R}$, Problem (Q_λ) satisfies a Fredholm alternative. State precisely what this means, along the same lines as in (b) and (c), and give a proof.

(e) Given $a_{ij}, b_i, c_i, d \in L^\infty(\Omega)$, for $i, j \in \{1, \ldots, N\}$, let $\bar{\bar{a}} := (a_{ij})_{i,j=1}^N$, $\bar{b} := (b_i)_{i=1}^N$, $\bar{c} := (c_i)_{i=1}^N$, and consider the following generalization of Problem (Q_λ): Given $\phi \in H^1(\Omega)^*$, find $u \in H^1(\Omega)$ such that

$$(\tilde{\mathrm{Q}}_\lambda) \qquad \int_\Omega \big\{ (\bar{\bar{a}} \nabla u + \bar{b} u) \cdot \nabla v + (\bar{c} \cdot \nabla u + d u + \lambda u) v \big\} = \phi(v) \quad \text{for all } v \in H^1(\Omega).$$

Assuming sufficient regularity of the coefficients, $(\tilde{\mathrm{Q}}_\lambda)$ is the weak version of a boundary-value problem $(\tilde{\mathrm{P}}_\lambda)$. Identify the problem $(\tilde{\mathrm{P}}_\lambda)$. Give sufficient conditions to guarantee that Problem $(\tilde{\mathrm{Q}}_\lambda)$ satisfies a Fredholm alternative. Skipping the technical details, explain how this is proved.

Problem 3

Assume that Ω is of class C^1 and let $a_{ij}, b_i \in C(\bar{\Omega})$ with $a_{ij} = a_{ji}$ for $i, j \in \{1, \ldots, N\}$. Suppose there exists $\eta \in (0, \infty)$ such that $\sum_{i,j=1}^N a_{ij}(x)\xi_i\xi_j \geq \eta|\xi|^2$ for all $x \in \bar{\Omega}$ and $\xi \in \mathbb{R}^N$. Define $L := -\sum_{i,j=1}^N a_{ij}\partial_{ij} + \sum_{i=1}^N b_i\partial_i$, let $u \in C^2(\Omega) \cap C(\bar{\Omega})$, and suppose that $Lu \geq 0$ in Ω.

(a) What are the assertions of the *weak maximum principle* and the *strong maximum principle*?

(b) Given $x_0 \in \partial\Omega$ such that $u(x) > u(x_0)$ for all $x \in \Omega$ and assuming that $u \in C^1(\bar{\Omega})$, what is the assertion of *Hopf's boundary-point lemma*?

(c) Given $f \in C(\Omega)$, $g \in C(\partial\Omega)$, show that the Dirichlet problem $Lv = f$ in Ω, $v = g$ on $\partial\Omega$ cannot have more than one solution in $C^2(\Omega) \cap C(\bar{\Omega})$. Also, show that any two solutions of the Neumann problem $Lv = f$ in Ω, $\frac{\partial v}{\partial \hat{n}} = g$ in $C^2(\Omega) \cap C^1(\bar{\Omega})$ differ by at most an additive constant.

Problem 4

(a) Let X be a Banach space, $S = (S_t)_{t \in \mathbb{R}_+}$ a family of bounded linear operators from X into itself. What do we mean when we say that S is a *strongly continuous contraction semigroup* on X? What do we mean by the *generator* of S?

(b) Assuming that Ω is of class C^2, let $X := L^2(\Omega)$ and define $A\colon D(A) \subset X \to X$ by $D(A) := H^2(\Omega) \cap H_0^1(\Omega)$ and $Au := \Delta u$ for $u \in D(A)$. Then A is the generator of a strongly continuous contraction semigroup S on X. Outline a proof. *Hint.* Recalling the Hille-Yosida theorem, what needs to be shown?

(c) Using the notation introduced in (b), let $u_0 \in D(A)$ and define $u\colon \mathbb{R}_+ \to X$ by $u(t) := S_t u_0$ for $t \in \mathbb{R}_+$. Explain in what sense u is a solution of the initial-boundary value problem

$$\begin{cases} u_t - \Delta u = 0 & \text{in } (0,\infty) \times \Omega, \\ u = 0 & \text{on } (0,\infty) \times \partial\Omega, \\ u = u_0 & \text{on } \{0\} \times \Omega. \end{cases}$$

Problem 5

Given $T \in (0, \infty)$, $f \in L^2((0,T) \times \Omega)$, $g \in H_0^1(\Omega)$, and $h \in L^2(\Omega)$, consider the initial-boundary value problem

$$\begin{cases} u_{tt} - \Delta u = f & \text{in } (0,T) \times \Omega, \\ u = 0 & \text{on } (0,T) \times \partial\Omega, \\ u = g,\ u_t = h & \text{on } \{0\} \times \Omega. \end{cases}$$

(a) Under the above conditions, the initial-boundary value problem has a unique weak solution u. In what sense? (Be precise!)

(b) Assuming sufficient regularity, u allows an expansion of the form $u(t,x) = \sum_{j=1}^\infty \alpha_j(t)\phi_j(x)$, for $(t,x) \in (0,T) \times \Omega$, where (ϕ_j) is an orthonormal basis of $L^2(\Omega)$, consisting of eigenfunctions of the Dirichlet problem $-\Delta u = \lambda u$ in Ω, $u = 0$ on $\partial\Omega$. Let (λ_j) be the corresponding sequence of eigenvalues. Identify the initial-value problems that determine the coefficients α_j.

(c) Suppose f is given by $f(t,x) := A(x)\cos(\omega t)$ for $(t,x) \in (0,\infty) \times \Omega$, with $A \in L^2(\Omega)$ and $\omega \in (0,\infty)$. What is the form of the coefficients α_j in this case? (Be specific!) What can you say about the behavior of u for large values of t if $\omega^2 = \lambda_n$ for some $n \in \mathbb{N}$? Give a physical interpretation of this phenomenon, for example, in terms of forced vibrations of a flexible membrane.