Mathematical Statistics Preliminary Examination

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August 21, 2012 9:00AM – 12:00PM

Solve any **five** problems out of the **seven** problems. The maximum possible score is 50 points. Each question is worth 10 points. If you work out more than five problems, your score is the sum of five highest points.

Name: ______________________________

Questions:

1. State and prove the Cramér-Rao inequality. What are two uses of this theorem?

2. Let $X_1, \dots, X_n$ be a random sample from the density $f_\theta(x) = \theta f_1(x) + (1-\theta) f_2(x)$, where f_1 and f_2 are two different densities and $\theta \in (0,1)$ is unknown.

   (a) Provide a necessary and sufficient condition for the likelihood equation to have a unique solution and show that if there is a solution, then it is the MLE of θ.

   (b) Derive the MLE of θ when the likelihood equation has no solution.

3. Suppose that T_n has a χ^2 distribution of n degrees of freedom. Show that $\sqrt{2T_n} - \sqrt{2n}$ converges in distribution to $N(0,1)$. Hint: A chi-squared random variable is the sum of squared standard Normal random variables.

4. Suppose that $X_1, \dots, X_n$ are independent and

$$P(X_i = k) = \frac{P(\text{Binomial}(m_i, \theta) = k)}{P(\text{Binomial}(m_i, \theta) > 0)} = \frac{\binom{m_i}{k} \theta^k (1-\theta)^{m_i - k}}{1 - (1-\theta)^{m_i}}$$

   for $k = 1, \dots, m_i$. The numbers $m_1, \dots, m_n$ are known integers with $m_i \geq 1$. That is, X_i has the conditional distribution of a Binomial(m_i, θ) random variable given that it is not 0.

   (a) Find a complete and sufficient statistic.

   (b) Assume that all the m_i are the same value m. Use the random variable $Y = I(X_i = m)$ to give a formula for the UMVUE of $\theta^m / [1 - (1-\theta)^m]$. You are not expected to simplify the formula.

   (c) For the case $n = 2$ and $m_1 = m_2 = m = 3$, what is the value of the estimate in the previous part if $X_1 = 3$ and $X_2 = 1$?

5. Suppose Y is a random variable with pdf $g(y)$, U is a random variable with a uniform$(0, 1)$ distribution, and Y and U are independent. Let $f(x)$ be a pdf. Suppose that there is a constant M such that $f(x) \le Mg(x)$ for all $x \in \mathbb{R}$. Define a random variable X as $X = Y$ if $U \le \dfrac{f(Y)}{Mg(Y)}$. Show that X has pdf $f(x)$.

6. Suppose the random variables C, T, and Z are related as follows:

$$P(T \ge t|Z) = e^{-Z\theta t}\,, \qquad P(C \ge c|Z) = e^{-Z\rho c}$$

and T and C are independent given Z. Assume further that Z has a gamma distribution with mean 1 and variance ϕ^{-1}; that is, it has the moment generating function $M_Z(t) = (1 - t/\phi)^{-\phi}$. Show that the joint distribution function of (T, C) can be found as

$$P(T \ge t, C \ge c) = \left(1 + \frac{\theta t}{\phi} + \frac{\rho c}{\phi}\right)^{-\phi}.$$

7. Find the likelihood ratio test (LRT) of

$$H_0 : \alpha \le 0 \qquad \text{vs} \qquad H_1 : \alpha > 0$$

based on a sample $X_1, \ldots, X_n$ from the two-parameter exponential distribution with pdf

$$f(x; \alpha, \beta) = \frac{1}{\beta} \exp\left(-\frac{x - \alpha}{\beta}\right) I_{[\alpha,\infty)}(x)$$

where both $-\infty < \alpha < \infty$ and $\beta > 0$ are unknown.