I R A M P
12(2), July 2021, pp. 67-72

ISSN: 2229-39-44

Bessonov Pulse Vector: On the Classification of Finite Energy Electromagnetic Waves

NIKOLAY L. POPOV* AND ALEXANDER V. VINOGRADOV

P.N. Lebedev Physical Institute, Leninsky Prospekt 53, 119991 Moscow, Russia; vinograd@sci.lebedev.ru
** Correspondence: popovnl@sci.lebedev.ru*

ABSTRACT: The time integrated strength of the electric field $\int_{-\infty}^{\infty} \boldsymbol{E}(\boldsymbol{r},t)dt = \boldsymbol{S_E}(\mathbf{r})$ of electromagnetic waves (EM) was widely studied, especially for unipolar, bipolar and few cycle EM pulses, after this parameter was suggested by E.G. Bessonov four decades ago. In this paper, it is shown that $\boldsymbol{S_E}(\boldsymbol{r}) = 0$ is a very general property of light considered as a free space solution to EM wave equations. It is valid for a wide class of EM pulses. This property can be useful in various applications of few cycle radiation and as a benchmark in EM and QED computations.

***Keywords*:** space–time couplings; spatiotemporal; ultrafast optics; unipolar pulses; few cycle pulses

1. INTRODUCTION

Few cycle lasers [1] are used in experiments nowadays. This necessitates theoretical investigations of new properties of free electromagnetic (EM) pulses [2-4]. For this purpose we will use the full set of electromagnetic wave equations under the practical condition that the energy of the electromagnetic pulse is finite. This requirement excludes monochromatic and plane waves, Gaussian beams, and some other frequently utilized models.

The distinctive property of finite energy EM pulses, which is clearly manifested and often discussed, is space–time couplings (STC). This implies that the spatial configuration of the field of the propagating pulse is constantly changing with time, while the time shape and the spectrum of the pulse change from point to point. For example, it is well known that when focusing, the shape and spectrum of the compressed pulses differ from the incident one, and it is not a matter of duration, but of the number of periods of the field. Any pulse of finite energy has the STC property. However, when the number of periods of the field N is large, these effects are weak and are not given importance. It is enough to characterize the structure of the pulse and its interaction with matter in terms of the average frequency ω and line width $\delta\omega$. If the pulse is short-period and N approaches 1, the values of ω and $\delta\omega$ become close. In this case, the result of the impact of a pulse on a substance essentially depends on its shape, and not on these average characteristics of the spectrum [5].

For a consistent description of STC effects, it is necessary to refer to models of finite energy pulses with nonseparable dependence on space and temporal coordinates. This can be achieved in exact solutions of Maxwell's equations, as well as in other rigorous methods. It should be noted here that, as a rule, finding exact solutions and studying their properties is also a difficult task [2,6–10]. On a qualitative level, we can say that STC is expressed in the extreme variability of any real electromagnetic pulse in time and space [11].

This work is devoted to another, in a sense, opposite property of electromagnetic pulses, which, like STC, is also important in ultrafast optics. This is due to the once rarely used characteristic of the EM field

$$\boldsymbol{S}_{\boldsymbol{E}}(\boldsymbol{r}) = \int_{-\infty}^{\infty} \boldsymbol{E}(\boldsymbol{r},t)dt, \tag{1}$$

which in the era of few cycle pulses, attracted increased attention. This value first appeared in the work [12] by E.G. Bessonov, who studied the radiation of particles in accelerators. To describe the radiation field $\boldsymbol{E}(\boldsymbol{r},t)$ of a system of charged particles, he introduced parameter (1) and proposed to use it to classify electromagnetic waves, calling waves with

$$\boldsymbol{S}_{\boldsymbol{E}}(\boldsymbol{r}) \neq 0 \tag{2}$$

strange. At the same time, as Bessonov showed, the radiation field of sources of a sufficiently wide class, namely, any system of charges performing a finite motion, satisfies the relation

$$\boldsymbol{S}_{\boldsymbol{E}}(\boldsymbol{r}) = 0. \tag{3}$$

He called EM waves, for which (3) is satisfied for all $\boldsymbol{r}$, usual. Thus, in the classification proposed by Bessonov, usual waves are necessarily bipolar. At the same time, strange waves, according to (2), can be both bipolar and unipolar (single sign). Their source can be charges that perform infinite motion, including bremsstrahlung, Compton scattering, the radiation of charged particles in bending magnets, the radiation of cosmic rays in the magnetic field of the Earth, radiation of electrons reflected from the surface of crystals, etc. [12]. The work of Bessonov received a noticeable response in accelerator and microwave communities. Since then, several theoretical and experimental papers on e-beam and other sources of bipolar and unipolar waves have been published (see [11] for references).

Interest in the topic increased sharply in the mid-1990s. The generation, application and study of unipolar pulses has become extremely relevant with the advent of the era of few cycle laser fields [13,14]. The main findings of [12] were again analyzed and confirmed [15,16]. The prospects and applications of unipolar pulses in microwave and optical ranges are presently being studied and reported in various new fields of science and technology. The parameter $\boldsymbol{S}_{\boldsymbol{E}}(\boldsymbol{r})$ is now also used in a broader sense than the criteria given by Bessonov formulas (1)–(3). For the characterization of bipolar strange waves, Arkhipov et al. [17] introduced the degree of unipolarity:

$$\xi(\boldsymbol{r}) = \frac{|\boldsymbol{S}_{\boldsymbol{E}}(\boldsymbol{r})|}{\int_{-\infty}^{+\infty}|\boldsymbol{E}(\boldsymbol{r},\boldsymbol{t})|dt} = \frac{\left|\int_{-\infty}^{+\infty}\boldsymbol{E}(\boldsymbol{r},\boldsymbol{t})dt\right|}{\int_{-\infty}^{+\infty}|\boldsymbol{E}(\boldsymbol{r},\boldsymbol{t})|dt}, \tag{4}$$

The method to measure the modulus of Bessonov vector (1) by observing quantum transitions induced by pulse interaction with a two-level system was proposed in [11].

In this work, it will be shown that in the absence of sources, any EM pulse is typically usual in the sense of Bessonov condition (3). In other words, condition (3) is the same inherent property of EM pulses in a vacuum as the invariants of energy, momentum, angular momentum, spin [2,18], the number of quanta [19–21] and STC, which were discussed above.

2. MATERIALS AND METHODS: SOME RELATIONS FOR THE VECTOR POTENTIAL IN K-SPACE

First, let's repeat that we are talking about impulses of a general form in free space. The only requirement is the finiteness of total energy.

As in the method of expansion of an EM field in terms of field oscillators [22], it is convenient to use the Fourier transform. This allows, instead of the field strengths $\boldsymbol{E}(\boldsymbol{r}, t)$ and $\boldsymbol{H}(\boldsymbol{r}, t)$, satisfying the free space Maxwell equations:

$$\begin{gathered}\mathrm{rot}\boldsymbol{E}(\boldsymbol{r},t) = -\frac{1}{c}\frac{\partial \boldsymbol{H}(\boldsymbol{r},t)}{\partial t},\ \mathrm{div}\boldsymbol{E}(\boldsymbol{r},t) = 0,\\ \mathrm{rot}\boldsymbol{H}(\boldsymbol{r},t) = \frac{1}{c}\frac{\partial \boldsymbol{E}(\boldsymbol{r},t)}{\partial t},\ \mathrm{div}\boldsymbol{H}(\boldsymbol{r},t) = 0.\end{gathered} \tag{5}$$

consider only vector potential $\boldsymbol{A}(\boldsymbol{r}, t)$, which satisfies the equations:

$$\begin{cases} \Delta \boldsymbol{A}(\boldsymbol{r},t) = \frac{1}{c^2}\ddot{\boldsymbol{A}}(\boldsymbol{r},t) \\ \mathrm{div}\boldsymbol{A}(\boldsymbol{r},t) = 0 \end{cases}, \tag{6}$$

where c is the speed of light in a vacuum. In this case, the vectors of the electric and magnetic fields are expressed in terms of $\boldsymbol{A}$ as follows:

$$\boldsymbol{E}(\boldsymbol{r},t) = -\frac{1}{c}\dot{\boldsymbol{A}}(\boldsymbol{r},t);\ \ \boldsymbol{H}(\boldsymbol{r},t) = \mathrm{rot}\,\boldsymbol{A}(\boldsymbol{r},t). \tag{7}$$

The transition to the Fourier space in Equations (6) after some transformations allows us to present the EM field as a set of independent harmonic oscillators. This is part of the standard procedure for quantizing the EM field. For the purposes of this work, it is sufficient to express the fields $\boldsymbol{E}(\boldsymbol{r}, t)$ and $\boldsymbol{H}(\boldsymbol{r}, t)$ in terms of the spatial harmonics of the vector potential $\boldsymbol{A}(\boldsymbol{k}, t)$; these fields depend on time according to the harmonic law. As a result, the proof of (3) is reducedto the study of integrals of rapidly oscillating functions.

For the Fourier harmonics of the vector potential $\boldsymbol{A}(\boldsymbol{k}, t)$, from Maxwell's equations in the form of (6), it is easy to obtain the following equations:

$$\ddot{\boldsymbol{A}}(\boldsymbol{k},t) + k^2c^2\boldsymbol{A}(\boldsymbol{k},t) = 0, \tag{8}$$

$$\boldsymbol{A}^*(\boldsymbol{k},t) = \boldsymbol{A}(-\boldsymbol{k},t), \tag{9}$$

$$\left(\boldsymbol{k}\cdot\boldsymbol{A}(\boldsymbol{k},t)\right) = 0. \tag{10}$$

Expressions for $\boldsymbol{E}(\boldsymbol{r}, t)$ and $\boldsymbol{H}(\boldsymbol{r}, t)$ are easily found from the relations (7):

$$\boldsymbol{E}(\boldsymbol{r},t) = -\frac{1}{c}\int \dot{\boldsymbol{A}}(\boldsymbol{k},t)\boldsymbol{e}^{i\boldsymbol{kr}}\boldsymbol{dk},\ \boldsymbol{H}(\boldsymbol{r},t) = i\int\left(\boldsymbol{k}\times\boldsymbol{A}(\boldsymbol{k},t)\right)e^{i\boldsymbol{kr}}\boldsymbol{dk},\ \boldsymbol{A}(\boldsymbol{r},t) = \int \boldsymbol{A}(\boldsymbol{k},t)e^{i\boldsymbol{kr}}\boldsymbol{dk}. \tag{11}$$

The total energy $\mathcal{E}$ does not depend on time:

$$\mathcal{E} = \int \epsilon(\boldsymbol{r},t)d\boldsymbol{r},\ \epsilon(\boldsymbol{r},t) = \frac{\boldsymbol{E}^2(\boldsymbol{r},t)+\boldsymbol{H}^2(\boldsymbol{r},t)}{8\pi}. \tag{12}$$

In $\boldsymbol{k}$-space expression (12), taking into account (11) and (9), (10) takes the form

$$\mathcal{E} = \pi^2\int[|\boldsymbol{E}(\boldsymbol{k},t)|^2 + |\boldsymbol{H}(\boldsymbol{k},t)|^2]d\vec{\boldsymbol{k}} = \frac{\pi^2}{c^2}\int\left[\left|\dot{\boldsymbol{A}}(\boldsymbol{k},t)\right|^2 + k^2c^2|\boldsymbol{A}(\boldsymbol{k},t)|^2\right]d\vec{\boldsymbol{k}}. \tag{13}$$

We now use equation (8), from which it obviously follows that

$$\boldsymbol{A}(\boldsymbol{k},t) = \boldsymbol{A}_-(\boldsymbol{k})\boldsymbol{e}^{-ickt} + \mathbf{A}_+(\boldsymbol{k})\boldsymbol{e}^{+ickt}, \tag{14}$$

and, therefore, the total energy (13) is equal to

$$\mathcal{E} = 2\pi^2\int k^2[|\boldsymbol{A}_-(\boldsymbol{k})|^2 + |\boldsymbol{A}_+(\boldsymbol{k})|^2]\boldsymbol{d}\vec{\boldsymbol{k}}. \tag{15}$$

The functions $\mathbf{A}_-(\boldsymbol{k})$ and $\mathbf{A}_+(\boldsymbol{k})$ are determined by the initial conditions for equation (8). For each of them, the transversality condition similar to (10) is satisfied, and, in addition, they are linked by the relation

$$\boldsymbol{A}_+(\boldsymbol{k}) = \boldsymbol{A}_-^*(-\boldsymbol{k}) \tag{16}$$

Thus, formula (14) expresses the vector potential of an arbitrary EM pulse of finite energy in terms of the vector functions $\mathbf{A}_-(\boldsymbol{k})$ and $\mathbf{A}_+(\boldsymbol{k})$, satisfying the transversality condition and being square-integrable with weight k^2. In the next section, this will be used to prove relation (3).

3. RESULTS: FREE EM PULSES ARE NOT STRANGE

Along with the strangeness vector $\boldsymbol{S}_E(\boldsymbol{r})$ (see (1)), consider the vector $\boldsymbol{S}_E(\boldsymbol{r}, T)$, where

$$\boldsymbol{S}_E(\boldsymbol{r}) = \lim_{T\to\infty} \boldsymbol{S}_E(\boldsymbol{r},T) = \lim_{T\to\infty} \int_{-T}^{T} \boldsymbol{E}(\boldsymbol{r},t)\,dt = -\frac{1}{c}\lim_{T\to\infty}[\boldsymbol{A}(\boldsymbol{r},T) - \boldsymbol{A}(\boldsymbol{r},-T)]. \tag{17}$$

For the first term in (17), taking into account the last formula of (11), as well as formula (14), we obtain

$$\boldsymbol{A}(\boldsymbol{r},T) = \int \boldsymbol{A}(\boldsymbol{k},T)e^{i\boldsymbol{kr}}\boldsymbol{dk} = \int \boldsymbol{A}_-(\boldsymbol{k})e^{i\boldsymbol{kr}-ickT}\boldsymbol{dk} + \int \boldsymbol{A}_+(\boldsymbol{k})e^{i\boldsymbol{kr}+ickT}\boldsymbol{dk}. \tag{18}$$

According to the Riemann–Lebesgue theorem [23], both terms in (18) disappear, as $T \to \infty$ for any $\boldsymbol{r}$. The condition for the applicability of this theorem is the convergence of the integrals

$$\int_0^\infty k^2 \mathrm{d}k \left|\int d\widehat{\Omega}\, \boldsymbol{A}_-(\boldsymbol{k})\, e^{i\boldsymbol{kr}}\right| \text{ and } \int_0^\infty k^2 \mathrm{d}k \left|\int d\widehat{\Omega}\, \boldsymbol{A}_+(\boldsymbol{k})\, e^{i\boldsymbol{kr}}\right|, \tag{19}$$

where $d\widehat{\Omega}$ means the integral over the angles of the vector $\boldsymbol{k}$. Taking into account the convergence of the energy integral (15), the requirement of convergence of the integrals in (19) does not seem excessively strict.

The second term on the right-hand side of (17) also disappears for $T \to \infty$. Thus, for a fairly large class of pulses in free space, we obtain

$$\boldsymbol{S}_{\boldsymbol{E}}(\boldsymbol{r}) = \lim_{T\to\infty} \int_{-T}^{T} \boldsymbol{E}(\boldsymbol{r},t)\,dt = 0, \tag{20}$$

and consequently, relation (3) is satisfied. Hence, we can summarize this section by saying that each of the field projections of the free EM pulse of a finite energy is typically a sign-variable function of time, the integral of which is zero at any point in space.

4. CONCLUSIONS AND DISCUSSION

Violations of relation (3) in the published exact solutions of the free Maxwell equations are not known to us (see [4,11,25]). However, we also failed to prove the convergence of integrals in (19) for an arbitrary electromagnetic pulse of finite energy. Therefore, the possibility of the existence of strange and unipolar pulses in a vacuum, although quite exotic, remains. This will happen if, despite the finiteness of the pulse energy (13), the conditions for the validity of the Riemann–Lebesgue theorem are broken; that is, the integral in (19) diverge. Note that the analysis of the radiation field of the laser medium in a cavity [14], and as was mentioned before the radiation of charges performing a finite motion [12], showed the validity of relation (3).

Table 1 shows the classification of EM waves in terms of relations (2) and (3), as well as the terminology introduced in Section 1. Table 1 yields the conclusion that rigorous modeling of unipolar (single sign) EM pulses and their applications requires accurate consideration of both the nature of the source and the propagation of the wave, since both factors affect the time shape of the pulse at the point of interest.

The 3rd column in Table 1, like the rest of the article, deals only with exact, finite-energy vector solutions to EM wave equations in free space. Important results beyond these restrictions (e.g., nondiffracting waves, scalar and 2D wave-packets, etc.) can be found in [26–29].

In conclusion, we note the coexistence of two properties inherent in EM pulses of finite energy. On the one hand, this is the extreme variability in space and time associated with STC, and on the other hand, the equality to zero of the S-function in the whole space as was proved here. The latter can be used to control the accuracy of EM and QED computations with finite energy pulses (compare with [30, 31]).

Table 1. Bessonov characteristic $S_E(r)$ and classification of EM pulses from moving charges (2nd column) and EM pulses travelling in free space (3rd column).

	EM Pulses from Moving Charges [12]			Free Space EM
Charge Motion	Bounded	Unbounded		pulses (This Paper)
$S_E(r)$	0	$\neq 0$	0	0
usual or strange	usual	strange	usual	usual
bipolar or unipolar	bipolar	both are possible	bipolar	bipolar

Acknowledgments: The authors are indebted to E. Oks for fruitful discussions.

References

1. Pickering, James David. *Ultrafast Lasers and Optics for Experimentalists*, IOP Publishing, 2021, DOI: 10.1088/978-0-7503-3659-8
2. Lekner, J. *Theory of Electromagnetic Pulses*; Morgan & Claypool Publishers: San Rafael, CA, USA, 2018.
3. Feng, S.; Winful, H.G.; Hellwarth, R.W. Spatiotemporal evolution of focused single-cycle electromagnetic pulses. *Phys. Rev. E* 1999, *59*, 4630.
4. April, A. Ultrashort Strongly Focused Laser Pulses in Free Space. In *Coherence and Ultrashort Pulse Laser Emission*; Duarte, F.J., Ed.; InTech: New York, NY, USA, 2010; pp. 355–382.
5. Keldysh, L.V. Multiphoton ionization by a very short pulse. *Phys. Usp*. 2017, *60*, 1187–1193.
6. Kiselev, A.P. Localized Light Waves: Paraxial and Exact Solutions of the Wave Equation (a Review). *Opt. Spectrosc*. 2007, *102*, 603–622.
7. Ziolkowski, R.W. LocalizedWaves: Historical and Personal Perspectives. In *Non-Diffracting Waves*; Hernández-Figueroa, H.E., Recami, E., Eds.; JohnWiley & Sons: New York, NY, USA, 2013; Chapter 2, pp. 69–81.
8. Artyukov, I.A.; Dyachkov, N.V.; Feshchenko, R.M.; Vinogradov, A.V. Energy density and spectrum of single-cycle and sub-cycle electromagnetic pulses. *Quantum Electron.* 2020 50, 187–194.
9. So, I.A.; Plachenov, A.B.; Kiselev, A.P. Simple unidirectional finite-energy pulses. *Phys. Rev. A* 2020, *102*, 063529.
10. Bialynicki-Birula, I. New solutions of the Dirac, Maxwell, and Weyl equations from the fractional Fourier transform. *Phys. Rev. D* 2021, *103*, 085001.
11. Popov, N.L.; Vinogradov, A.V. Space-Time Coupling: Current Concept and Two Examples from Ultrafast Optics Studied Using Exact Solution of EM Equations. *Symmetry* 2021, *13*, 529.
12. Bessonov, E.G. On a class of electromagnetic waves. *Zh. Eksp. Teor. Fiz*. 1981, *80*, 852–858.
13. Kaplan, A.E.; Shkolnikov, P.L. Electromagnetic "Bubbles" and Shock Waves: Unipolar, Nonoscillating EM Solitons. *Phys. Rev. Lett. D*, 1995, 2316-2319.
14. Miloševi'c, D.B.; Paulus, G.G.; Bauer, D.; Becker, W.J. Above-threshold ionization by few-cycle pulses. *Phys. B: At. Mol. Opt. Phys*. 2006, ***39***, R203–R262.
15. Kim, K.J.; McDonald, K.T.; Stupakov, G.V.; Zolotorev, M.S. A bounded source cannot emit a unipolar electromagnetic wave. *arXiv* 2000, arXiv:physics/0003064.
16. Kim, K.J.; McDonald, K.T.; Stupakov, G.V.; Zolotorev, M.S. Comment on "Coherent Acceleration by Subcycle Laser Pulses". *Phys. Rev. Lett*. 2000, *84*, 3210.
17. Arkhipov, R.M.; Pakhomov, A.V.; Arkhipov, M.V.; Babushkin, I.; Tolmachev, Y.A.; Rosanov, N.N. Generation of unipolar pulses in nonlinear media. *JETP Lett*. 2017, *105*, 408–418.
18. Feshchenko, R.M.; Vinogradov, A.V. On the number and spin of photons in classical electromagnetic fields. *Phys. Scr.* 2019, *94*, 065501.
19. Zeldovich, Y B. Number of quanta as an invariant of the classical electromagnetic field. *Sov. Phys.—Dokl*. 1966, *10*, 771.
20. Bialynicki-Birula, I. Photon Wave Number. In *Progress in Optics*; Wolf, E., Ed.; Elsevier Science: New York, NY, USA, 1996; Volume 36, pp. 245–294.
21. Feshchenko, R.M.; Vinogradov, A.V. On the number of photons in a classical electromagnetic field. *J. Exp. Theor. Phys*. 2018, *127*, 274–278.
22. Landau, L.D.; Lifshitz, E.M. *The Classical Theory of Fields*, 2nd ed.; Addison-Wesley, Reading, MA, USA, 1962.
23. Olver, F. *Asymptotics and Special Functions*; Academic Press: New York, NY, USA, 1974.
24. Zangwill, A. *Modern Electrodynamics*; Cambridge University Press: Cambridge, UK, 2012.
25. Ivan Gonoskov, Andrea Aiello, Simon Heugel and Gerd Leuchs, Dipole pulse theory: Maximizing the field amplitude from 4π focused

laser pulses, *Phys. Rev. A* 2014, *86*, 053836.

26. Kondakci, E.; Abouraddy, A.F. Diffraction-free space-time light sheets, *Nat. Photon.* 2017, *11*, 733.
27. Besieris, I.; Abdel-Rahman, M.; Shaarawi, A.; Chatzipetros, A. Two fundamental representations of localized pulse solutions to the scalar wave equation. *Prog. Electromagn Res.* 1998, *19*, 981–984.
28. Fagerholm, S.J.; Friberg, A.T.; Salomaa, M.M., Unified description of nondiffracting X and Y waves, *Phys. Rev. E* 2000 62, 4261.
29. Saari; Reivelt, K. Generation and classification of localized waves by Lorentz transformations in Fourier space. *Phys. Rev. E* 2004**,** *69*, 036612.
30. Madsen, L.B. Gauge invariance in the interaction between atoms and few-cycle laser pulses. *Phys. Rev. A* 2002, *65*, 053417.
31. Fedotov A.M.; Korolev, K.Yu.; Legkov, M.V. Exact analytical expression for the electromagnetic field in a focused laser beam or pulse. *Proc. SPIE* 2007, *6726*, 672613.