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Self-similar Analytic Solutions to Resonance Line Radiation Transport in a Plasma Slab and Comparison with Escape Probability Method

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ABSTRACT: The present paper extends and applies the former analysis of the possibility to obtain the analytical solutions of the equation of nonlocal (superdiffusive) radiative transfer in resonance atomic/ionic lines in the Biberman-Holstein model, which might be used for testing the edge plasma codes. That analysis has shown that for some types of similarity of spatial profiles of three characteristics, namely, background plasma density, line shape width and non-radiation source of atomic excitation, the profile of excited atoms density may be described analytically in terms of the similarity of the above-mentioned profiles. Here we present an example of application of this approach to the 3D transport of the Ly-alpha radiation in an inhomogeneous hydrogen plasma slab for the given profiles of main parameters of plasma, which are compatible with global transport problem. The potential of the available exact analytic solution is demonstrated via comparing the obtained analytic solution with the respective approximate solution in the framework of the escape probability method.

1. INTRODUCTION

The representation of kinetic equations in the self-similar variables allows one to obtain analytic solutions, which may be very helpful for testing the respective blocks of numerical codes for transport phenomena. The examples include the steady-state collisional kinetic equations for electrons [1] and neutral atoms [2] in a strongly inhomogeneous plasma. Here, self-similarity appears to be applicable to the cases of nonlocal (non-diffusive) correlations of the distribution function, that includes the cases of superthermal electrons [1] and fast neutrals produced by the charge exchange [2]. Another type of self-similarity was found [3] for the non-steady-state Biberman-Holstein (B-H) equation (see, *e.g.*, [4,5]) for radiative transfer in resonance atomic/ionic lines in a homogeneous media. Here again the self-similarity is expressed in terms of characteristics of nonlocality of the B-H radiative transfer.

The approach [1, 2] was extended in [6] to the case of the 1D transport of resonance line radiation in the framework of the steady-state Biberman-Holstein equation. It was shown that for some types of similarity of spatial profiles of three characteristics, namely, background plasma density, line shape width and non-radiation source of atomic excitation, the profile of excited atoms density may be described analytically in terms of the similarity of the above-mentioned profiles. The revealed cases of analytical solutions were suggested for testing the radiative transfer codes in edge plasmas, including the codes for radiative transfer in a background plasma and for radiation losses by an impurity in plasmas.

Here we extend the formalism [6] to the case of the 3D transport of the Ly-alpha radiation in an inhomogeneous hydrogen plasma slab and apply the results for the given profiles of main parameters of plasma, which are compatible with global transport problem. The potential of the available exact analytic solution is demonstrated via comparing the obtained analytic solution with the respective approximate solution in the framework of the escape probability method (see, *e.g.*, [7, 8]).

2. SELF-SIMILARITY OF THE BIBERMAN-HOLSTEIN EQUATION IN INHOMOGENEOUS MEDIA

Here we extend the formalism [6] to the case of the 3D transport of the Ly-alpha radiation in an inhomogeneous hydrogen plasma slab of the thickness L.

The Biberman-Holstein (B-H) equation for radiative transfer in resonance atomic/ionic lines in an inhomogeneous media in a three-dimensional steady-state case in the case of isotropic radiators when probabilities of emission of a photon and the corresponding cross-section of absorption in the rest frame of the atom don't depend on the direction of a photon, has the following form (cf. [4, 5]):

$$\int_0^\infty d\omega\, k(\omega,z)\int_0^L dz_1 \frac{n(z_1)P(\omega,z)}{4\pi\tau}\int_0^\infty \frac{2\pi\rho_1 d\rho_1}{(z-z_1)^2+\rho_1^2}\times\exp\left(-\sqrt{1+\frac{\rho_1^2}{(z-z_1)^2}}\left|\int_z^{z_1} k(\omega,z_2)\,dz_2\right|\right)+q(z)=\left(\frac{1}{\tau}+w(z)\right)n(z), \quad (1)$$

where we choose a variable z as a linear coordinate across the slab. $n(z)$ is the density of excited atoms, $P(\omega, z)$ is the (normalized over frequency ω) line shape of the photon emission by an excited atom at the point z, $k(\omega, z)$ is the coefficient of absorption of a photon by the excited atom (*i.e.* inverse free path of the photon) at the point z; τ is the mean lifetime of the atom's excited state in the case of no quenching (so called, lifetime with respect to spontaneous radiative decay); $q(z)$ is the source of excitation of atoms by all processes, except absorption of photons, $w(z)$ is the sink function (*i.e.* the rate of depopulation of the excited state by all processes, except spontaneous radiative decay). Integral terms consider the emission of a photon at a distant point z_1 and the absorption at the current point z taking into account the possibility of absorption by the other atoms on this way.

The application of the analytic solution [6] is most relevant in the case of plasmas which, in the space of main plasma parameters and kinetic conditions, lie rather far from the kinetic limits — the local thermodynamical equilibrium (LTE) (*i.e.* when the distribution over excited states is very close to the Boltzmann distribution and, for calculating the radiation losses, there is no need to solve the B-H equation) and the coronal model (*i.e.* when the kinetics of excited state populating is very simple, and the numerical simulations of the unified kinetic problem are much less complicated as compared to the intermediate between the LTE and the coronal limit). In the intermediate kinetic states, the simulations of the radiative transfer problem with allowance for all sources of non-equilibrium (*e.g.* strong influx of the neutral atoms from the boundary, strong coupling of radiative transfer to the transport of particles, etc.) there is a large freedom in choosing the examples for testing the approach developed in [6]. Indeed, in the intermediate between the LTE and the coronal limit, the populating and depopulating of the excited states takes place via the cascade processes which exceed the respective contributions in the coronal limit. Here we assume that

(i) one can introduce a single parameter which describes the net effect of the cascades on populating and depopulating of the excited state $n = 2$ responsible for the Ly-alpha radiative transport, and
(ii) this parameter may be taken equal to a constant, C_{cascade}, *i.e.* not dependent on the local plasma parameters (temperature, density, boundary conditions, etc.) in the selected range of plasma parameters (*e.g.*, for electron density $10^{13} < n_e < 10^{16}$ cm^{-3} and electron temperature $20 < T_e < 200$ eV, see below). The choice of the constant should agree with the known data (*e.g.* in [9]) for the excited state population in the non-LTE plasma without radiative transfer.

This gives:

$$w(z) = \nu_{\text{quench}}(z), \quad (2)$$

$$q(z) = C_{\text{cascade}} n_0(z) \frac{g_1}{g_0} w(z) \exp\left(-\frac{\Delta E}{T_{\text{e}}(z)}\right). \tag{3}$$

Here ν_{quench} is the frequency of quenching by electron impact, *i.e.* the excited atom's inverse lifetime with respect to this process. Following [5], let us introduce ν_{quench} in the form

$$\nu_{\text{quench}}(z) = n_{\text{e}}(z)\langle v\sigma_{01}\rangle(z)\left(n_0(z)/n_B(z)\right), \tag{4}$$

where $<v\sigma_{01}>(z)$ is the atom excitation rate due to electron impact [9],

$$n_B(z) = n_0(z) \frac{g_1}{g_0} \exp\left(-\frac{\Delta E}{T_{\text{e}}(z)}\right), \tag{5}$$

is "Boltzmann" excited level population, g_1 and g_0 are statistical weights, $\Delta E = E_1 - E_0$ is the transition energy, $n_0(z)$ is the density of atoms in the ground state.

The absorption coefficient (without stimulated emission) can be written as:

$$k(\omega, z) = B_{01} \cdot n_0(z) \cdot \frac{\hbar\omega_0\, \varphi(\omega, z)}{4\pi} = C_{01} \cdot n_0(z) \cdot \varphi(\omega, z), \tag{6}$$

where ω_0 is the frequency corresponding to the center of the spectral line, B_{01} is the Einstein coefficient (for absorption), $\varphi(\omega, z)$ is the (normalized over frequency) line shape of the photon absorption, $C_{01} = B_{01}\, \hbar\omega_0/4\pi$.

We assume the line shape to have the following form

$$P(\omega, z) = \varphi(\omega, z) = \frac{1}{\omega_T(z)} a\left(\frac{\omega}{\omega_T(z)}\right), \tag{7}$$

where $\omega_T(z)$ is the line shape width.

In some cases, the transfer equations allow substantial simplification. In [1, 2] it was shown that, under condition λ/S = const, where λ is the mean free path, S is the characteristic scale length of variation of the distribution function (see [1, 2] for λ and S definitions), it is possible to introduce the self-similar variables allowing to find an exact analytical solution of the kinetic equations.

We will try to proceed in a similar way and look for the solution in the form

$$n(z) = (\tilde{\omega}_T)^{\alpha} \cdot \frac{n_0(z)}{(\omega_T(z))^{\alpha}}, \ \tilde{\omega}_T = \omega_T(L/2) = \text{const}, \tag{8}$$

where α is the adjustable parameter, $\tilde{\omega}_T$ is the spectral line width in the center of a slab.

Here the condition λ_{ph}/S = const can be rewritten as

$$\frac{\omega_T(z)}{n_0(z)} \frac{d \ln \omega_T(z)}{dz} \equiv \gamma = \text{const}, \tag{9}$$

assuming monotonic behavior of the profile $\omega_T(x)$.

Similarly to [6], under the above assumptions one can introduce self-similar variables

$$\xi(\omega)=\frac{\omega}{\omega_T(x)},\ \eta(\omega_T')=\frac{\omega}{\omega_T'},\ \sigma(\omega_T'')=\frac{\omega}{\omega_T''}, \tag{10}$$

where $\omega_T'\equiv\omega_T\left(z_1\right),\ \omega_T''\equiv\omega_T\left(z_2\right)$. One can see that after simple transformations Eq. (1) differs from the equation in [6] only in the fact that there will be an exponential integral $E_1(z)$ in it instead of ordinary exponent. It follows that the formalism [6] works here, however Eq. (1) may be transformed to a similar form:

$$\left(1+\tau w(z)\right)=J_{\text{slab}}^{(3\text{D})}(z,\alpha,\beta)+q(z)\cdot\frac{\tau}{(\tilde{\omega}_T)^{\alpha}}\cdot\frac{(\omega_T(z))^{\alpha}}{n_0(z)}, \tag{11}$$

$$J_{\text{slab}}^{(3\text{D})}(z,\alpha,\beta)=\frac{\beta}{2}\int_{-\infty}^{\infty}d\xi\int_{\frac{\omega_T(z)}{\omega_T(L)}\xi}^{\frac{\omega_T(z)}{\omega_T(0)}\xi}d\eta\frac{1}{\eta}\left(\frac{\eta}{\xi}\right)^{\alpha}a(\eta)a(\xi)E_1\left(-\beta\left|\int_{\xi}^{\eta}\frac{1}{\sigma}a(\sigma)d\sigma\right|\right), \tag{12}$$

Here we introduced the dimensionless parameter $\beta\equiv C_{01}/\gamma$, which is related to optical depth Θ:

$$\Theta=\int_0^L k(\omega_0,z)dz=C_{01}\int_0^L\frac{n_0(z)}{\omega_T(z)}a(0)dz,\ \beta=\frac{1}{a(0)}\left[\ln\left(\frac{\omega_T(L)}{\omega_T(0)}\right)\right]^{-1}\Theta. \tag{13}$$

Function $J(z,\alpha,\beta)$ tends to a constant equal 1 with $\beta\to\infty$ (large optical depth). This limit, as it can be proved in general case, corresponds to a full compensation of absorption and emission of photons at a given point (the latter is valid *e.g.* in the infinite media).

Using Eq. (11) and taking Eq. (8) into account, we obtain

$$n(z)=n_0(z)\cdot\frac{C_{\text{cascade}}\tau w(z)\dfrac{g_1}{g_0}\exp\left(-\dfrac{\Delta E}{T_e(z)}\right)}{\left(1+\tau w(z)\right)-J(z,\alpha,\beta)}. \tag{14}$$

While the population as a function of z is obtained, one can estimate the radiative energy flux density in one direction [10].

The restriction of the formalism (8) on the class of possible profiles of temperatures and densities takes the following form in the case under consideration:

$$\frac{C_{\text{cascade}}\tau\nu_{\text{quench}}(z)\dfrac{g_1}{g_0}\exp\left(-\dfrac{\Delta E}{T_e(z)}\right)}{\tau\nu_{\text{quench}}(z)+1-J(z,\alpha,\beta)}=\frac{(\tilde{\omega}_T)^{\alpha}}{(\omega_T(z))^{\alpha}}. \tag{15}$$

One can see that profile of density of atoms in the ground state $n_0(z)$ drops out from this condition, and the restriction determines the relation between the profiles of electron density $n_e(z)$ and temperature $T_e(z)$, which define the populating and depopulating rates. Let us explicit quenching $\tau\nu_{\text{quench}}(z)$ from (15)

$$\tau\nu_{\text{quench}}(z)=\frac{1-J(z,\alpha,\beta)}{\dfrac{(\omega_T(z))^{\alpha}}{(\tilde{\omega}_T)^{\alpha}}C_{\text{cascade}}\dfrac{g_1}{g_0}\exp\left(-\dfrac{\Delta E}{T_{\text{e}}(z)}\right)-1}. \tag{16}$$

On the other hand, we have Eq. (4) for $\nu_{\text{quench}}(z)$. Hence, we obtain $n_{\text{e}}(z)$ profile as follows

$$n_{\text{e}}(z)=\frac{1-J(z,\alpha,\beta)}{\dfrac{(\omega_T(z))^{\alpha}}{(\tilde{\omega}_T)^{\alpha}}C_{\text{cascade}}\dfrac{g_1}{g_0}\exp\left(-\dfrac{\Delta E}{T_{\text{e}}(z)}\right)-1}\cdot\frac{1}{\tau\langle v\sigma_{01}\rangle(z)\dfrac{g_0}{g_1}\exp\left(\dfrac{\Delta E}{T_{\text{e}}(z)}\right)}. \tag{17}$$

So, Eq. (15) defines the relation between $n_e(z)$ and $T_e(z)$.

Thus, with (14) and (15) taken into account, the profile of the excited atoms density takes the following form

$$n(z)=n_0(z)\cdot\frac{C_{\text{cascade}}\tau\nu_{\text{quench}}(z)\dfrac{g_1}{g_0}\exp\left(-\dfrac{\Delta E}{T_{\text{e}}(z)}\right)}{\tau\nu_{\text{quench}}(z)+1-J(z,\alpha,\beta)}. \tag{18}$$

3. APPLICATION TO LYMAN-ALPHA RADIATION TRANSPORT

To prove the solutions obtained in the previous section and to estimate if the plasma parameters' profiles can satisfy the requirements imposed by the model, let us consider the following testing problem.

We will consider the 3D transport of the Ly-alpha radiation in an inhomogeneous hydrogen plasma slab of the thickness L = 100 cm. Let us assume the linear profiles both for the temperature of atoms $T_a(z)$ and for the electron temperature $T_e(z)$:

$$T_a(z) = 1.4z + 25, \tag{19}$$

$$T_e(z) = 1.4z + 20 \tag{20}$$

Temperatures $T_a(z)$ and $T_e(z)$ are in electron volts. The profile of the density of atoms in the ground state $n_0(z)$ is obtained from the condition γ = const (9), so

$$n_0(z)=\frac{1}{\gamma}\omega_T(z)\frac{d\ln\omega_T(z)}{dz}. \tag{21}$$

Let us consider the Doppler spectral line shape:

$$a_{\text{Doppler}}\left(\frac{\omega}{\omega_T(z)}\right)=\frac{1}{\sqrt{\pi}}\exp\left[-\left(\frac{\omega}{\omega_T(z)}\right)^2\right],\ \omega_T(z)=\frac{\omega_0}{c}\sqrt{\frac{2T_{\text{a}}(z)}{m}}, \tag{22}$$

where m is atom's mass. Therefore for Doppler line shape, Eq. (21) takes the following form

$$n_0(z)=\frac{\omega_0}{c}\sqrt{\frac{1}{2mT_{\text{a}}(z)}}\frac{1}{\gamma}\frac{dT_{\text{a}}(z)}{dz}. \tag{23}$$

Let us choose such value of γ that β = 150, *i.e.* $\gamma = C_{01}/150$ because of $\beta = C_{01}/\gamma$.

The optical thickness in case of temperatures (19) and β = 150 is about Θ = 80.

One can obtain the electron density profile $n_e(z)$ from the condition (17). In case of Doppler line shape

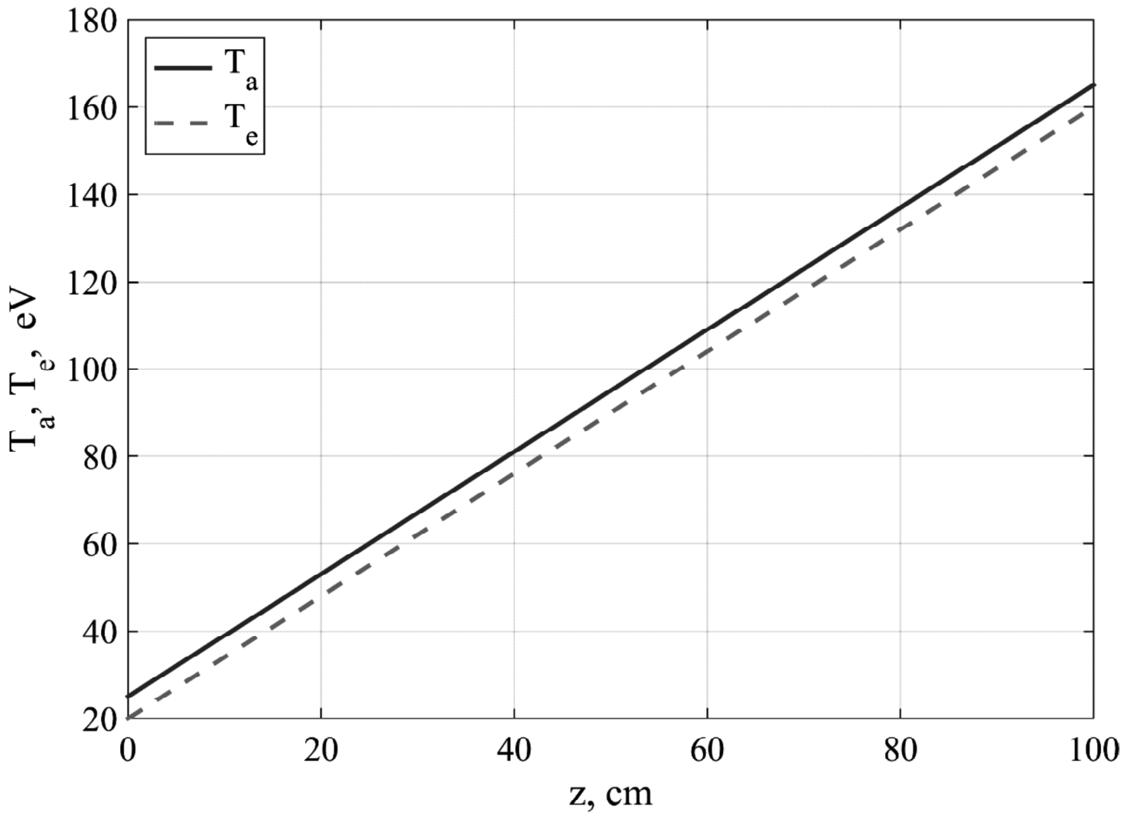
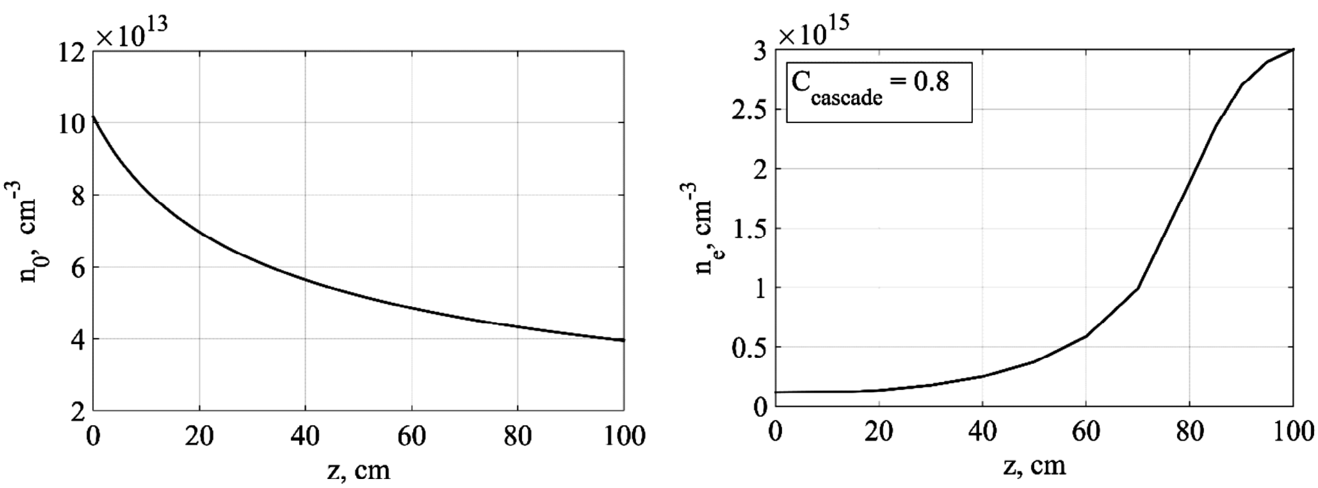


Figure 1: The profiles of the atom temperature $T_a(z)$ (19) and electron temperature $T_e(z)$ (20).



(a) **(b)**

Figure 2: (a) The profile of the density of atoms in the ground state $n_0(z)$ (23); (b) the electron density profile $n_e(z)$ (24) with α = –2.

$$n_{\mathrm{e}}(z)=\frac{1-J(z,\alpha,\beta)}{\left(\dfrac{T_{\mathrm{a}}(z)}{T_{\mathrm{a}}(L/2)}\right)^{\alpha/2}C_{\mathrm{cascade}}\dfrac{g_1}{g_0}\exp\left(-\dfrac{\Delta E}{T_{\mathrm{e}}(z)}\right)-1}\cdot\frac{1}{\tau\langle v\sigma_{01}\rangle\left(T_{\mathrm{e}}(z)\right)\dfrac{g_0}{g_1}\exp\left(\dfrac{\Delta E}{T_{\mathrm{e}}(z)}\right)}. \tag{24}$$

Here $\Delta E \approx 10, 2$ eV (Ly-alpha), $\tau = 10^{-9}$s.

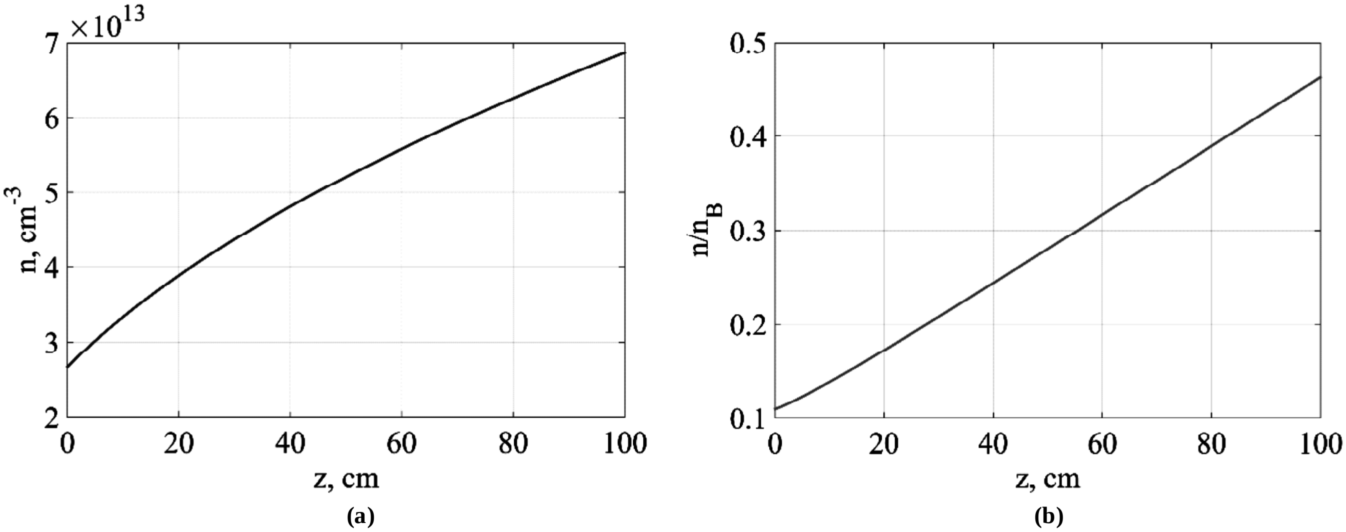


Figure 3: (a) The profile of the excited atoms density $n(z)$ (18) with (19)-(24) taken into account; (b) the ratio $n_1(z)/n_B(z)$.

4. COMPARISON WITH ESCAPE PROBABILITY METHOD

To demonstrate the potential of the available exact analytic solution let us compare the obtained analytic solution with the respective approximate solution in the framework of the escape probability method (see, *e.g.*, [5, 7, 8]). According to this method the excited atoms density profile can be expressed in the following form:

$$n_{\mathrm{EP}}(z)=\frac{q(z)}{w(z)+\frac{1}{\tau}\bar{T}(z)}. \qquad (25)$$

Here $\bar{T}(z)$ is the Holstein functional averaged over solid angles (cf. *e.g.* [5]), *i.e.* $\bar{T}(z)$ is the probability of the photon to escape the media volume without any acts of absorption/reemission:

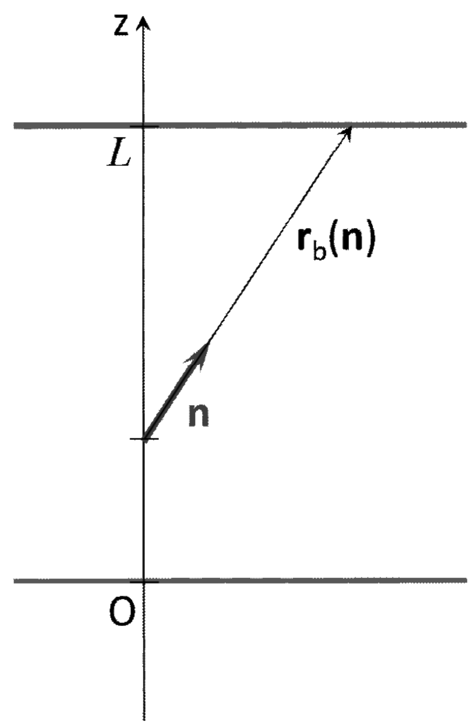


Figure 4: Geometrical variables describing $\bar{T}$ (26).

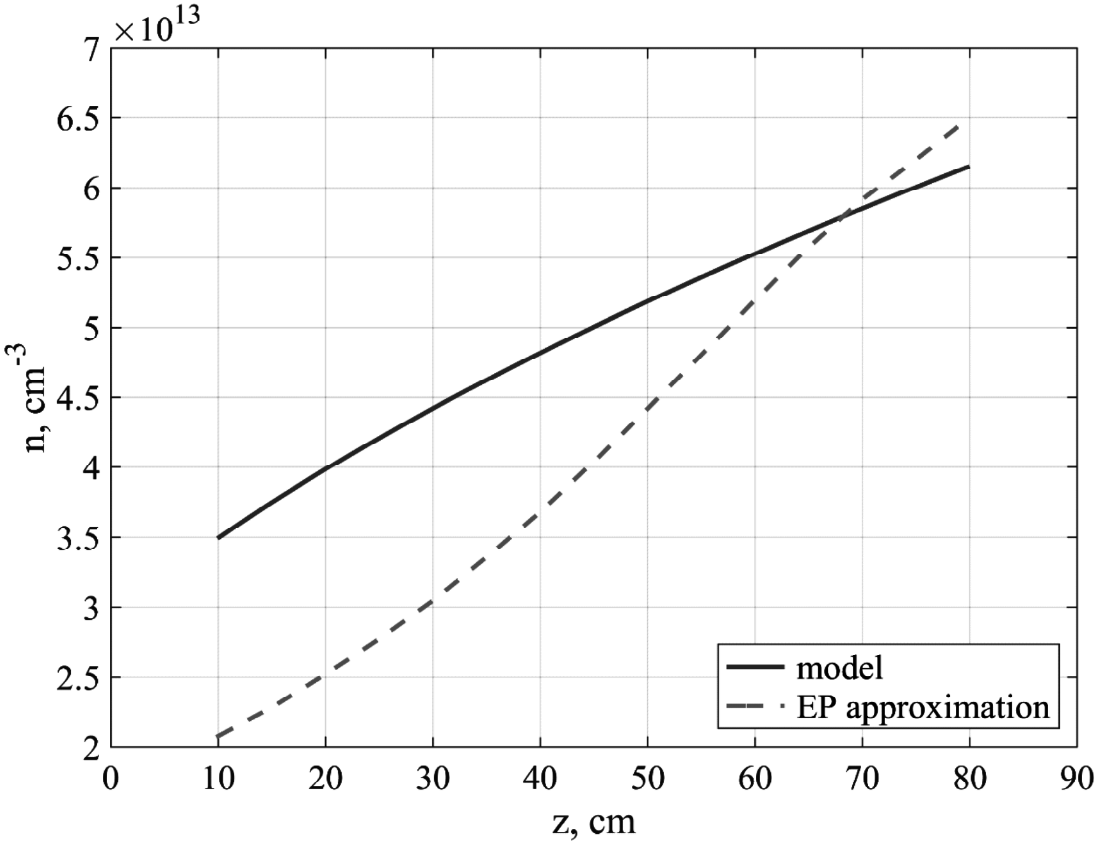


Figure 5: The comparison of results of the excited atoms density calculations by Eq. (18) (solid violet curve) and by escape probability method (25) (dashed green curve).

$$\overline{T}(r)=\frac{1}{4\pi}\int\limits_{(4\pi)} d\Omega_{\mathbf{n}}\int\limits_{0}^{\infty} d\omega P(\omega,r)\exp\left\{-\int\limits_{\mathbf{r}}^{\mathbf{r}_b(\mathbf{n})} k(\omega,r')ndr'\right\}. \quad (26)$$

Geometrical variables describing $\overline{T}$(26) are shown in the Figure 4.

The results of the excited atoms density calculations by Eqs. (18) and (25) are compared in the Figure 5.

The results of comparison in Figure 5 show that in the region of applicability of both approaches, namely not close to the boundary of the plasma slab, the EP approximation is rather good in the region of the hotter plasma. However, in the region of the lower temperature the EP approximation is less accurate. This may be explained in the following way: in such an inhomogeneous plasma of not so high optical depth, the applicability of the EP method (namely, the radiative exchange of atom excitation between closest neighbors should be compensated with a high accuracy) may be not guaranteed.

5. CONCLUSIONS

The extension of the formalism [6] to the more realistic case, namely, the 3D transport of the Ly-alpha radiation in an inhomogeneous hydrogen plasma slab enabled us to apply the results for the given profiles of main parameters of plasma, which are compatible with global transport problem. To comply with the latter feature, we take a simplified model of populating and depopulating the excited state by all the non-radiative processes, which allows analyzing the main features of the radiative transfer in a broad range of intermediate states between the local thermodynamic equilibrium (LTE) and the coronal limit. The potential of the available exact analytic solution is demonstrated via comparing the obtained analytic solution with the respective approximate solution in the framework of the escape probability (EP) method. This comparison allowed the direct estimation of the accuracy of the EP method in the range of applicability of the both approaches, namely, in the entire space volume except for the peripheral regions close to the boundaries.

In summary, we would like to note that the method, developed in [6] and the present paper for deriving the analytic solutions for nonlocal (superdiffusive) transport of resonance radiation in laboratory and astrophysical plasmas, provides a tool for testing both the computational elements of the complicated codes and the analytic and semi-analytic approximations used in these codes.

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REFERENCES

[1] S.I. Krasheninnikov, Sov. Phys. JETP **67** (1988) 2483.

[2] P. Helander, S.I. Krasheninnikov, Phys. Plasmas **3** (1996) 226.

[3] A.B. Kukushkin, P.A. Sdvizhenskii. J. Phys. A: Math. Theor. **49** (2016) 255002 (13 pp)

A.B. Kukushkin, P.A. Sdvizhenskii.Proc. 41st EPS Conf. Plasma Physics, Berlin, 2014, P4.133.

[4] Biberman L.M., Vorob'ev V.S., Yakubov I.T. Kinetics of Nonequilibrium Low Temperature Plasmas (New York: Consultants Bureau, 1987).

[5] Abramov V.A., Kogan V.I., Lisitsa V.S., In: *Reviews of Plasma Physics*, 1987, Vol.12 (Leontovich M.A., Kadomtsev B.B., Eds.) New York: Consultants Bureau, p.151.

[6] Sdvizhenskii P.A., Krasheninnikov S.I., Kukushkin A.B., Contributions to Plasma Physics (CPP) 56, 669-674, 2016, DOI 10.1002/ctpp.201610058

[7] Napartovich A. P., Teplofiz. Vys. Temp. 9, 26 (1971).

[8] Kalkofen W. In: Methods in Radiative Transfer (Kalkofen W., Ed.), Cambridge: Cambridge Univ. Press, 1984, chapter 1.

[9] Sobelman I.I., Vainshtein L.A., Yukov E.A. *Excitation of Atoms and Broadening of Spectral Lines*, 2nd Ed. Springer, 1995. 310 p.

[10] Sdvizhenskii P.A., Krasheninnikov S.I., Kukushkin A.B., Proc. 43rd EPS Conference on Plasma Phys., Leuven, Belgium, 4-8 July 2016, Europhysics Conference Abstracts, ECA vol. **40**, P2.026.