ALGEBRA PRELIMINARY EXAMINATION

September 29, 2007

I. Groups and Modules

Do problems 1 and 2 and any two of the remaining three. Here, and throughout the exam, R is a ring with identity $1 \neq 0$, and all R-modules are unitary left R-modules.

1. Prove the First Sylow Theorem; that is, if p^n is the largest power of the prime p that divides the order of a finite group G, prove that G has a subgroup of order p^n.

2. Suppose that G is a finite simple group of order 60. If G contains a subgroup of index 5, show that $G \cong A_5$.

3. Is every group of order 2007 solvable? Why or why not?

4. Show that $\mathbb{Q}$ is not a free $\mathbb{Z}$-module.

5. Prove that every R-module is projective if and only if every R-module is injective.

II. Rings, Modules, and Galois Theory

Do problems 6 and 7 and any two of the remaining three.

6. Determine the Galois groups of the following polynomials over $\mathbb{Q}$.
   (a) $f(x) = (x^2 - 2)(x^2 + 1)$.
   (b) $g(x) = x^4 + x^3 + x^2 + x + 1$.
   (c) $h(x) = x^3 - x - 1$.

7. Sketch a proof that every PID is a UFD. Show, by example, that not every UFD is a PID.

8. If R is a commutative ring, and if I is a proper ideal, then any ideal $J \neq R$ of R maximal with respect to containing I is prime.

9. Let $J(R) = \bigcap\{M : M \text{ is a maximal ideal of } R\}$. (You can assume that R is commutative if you wish.)
   (a) Show that if $r \in J(R)$, then $1 + r$ and $1 - r$ are units of R.
   (b) Use part (a) to show that if K is a finitely generated R-module such that $J(R) \cdot K = K$, then $K = 0$.

10. Given a tower of fields

$$K = E_0 \leq E_1 \leq E_2 \leq \cdots \leq E_n = F$$

such that E_j is (finite dimensional) Galois over E_{j-1} for each j, show that F is Galois over K.