ALGEBRA PRELIMINARY EXAMINATION

May 21, 2005

I. Groups

Do problems 1 and 2 and any two of the remaining four.

1. State and prove the Second Isomorphism Theorem.

2. Construct (up to isomorphism) all abelian groups of order 500. Where in your list is $\mathbb{Z}_{500}$? Explain your answer.

3. Prove each of the following statements.
   (a) A group of order 2005 is not simple.
   (b) A group of order p^2, p a prime, is abelian.

4. Let a and b be elements in a group G of finite orders m and n respectively. Prove that if m and n are relatively prime, then the order of ab is mn.

5. Recall that the center $Z(G)$ of a group G is defined by

$$Z(G) = \{g \in G : ga = ag \text{ for all } a \in G\}.$$

   Prove that $Z(G)$ is a normal subgroup of G and that if $G/Z(G)$ is cyclic, then G is abelian.

6. State Cauchy's Theorem and prove it without using any of the Sylow Theorems.

II. Rings and Modules

Do problems 7 and 8 and any two of the remaining four. Throughout this section, R denotes a ring with identity $1 \neq 0$ and all R-modules are unitary left R-modules.

7. Show that every nonzero prime ideal of a PID is maximal.

8. Regard $\mathbb{Q}$ and the polynomial rings $\mathbb{Z}[x]$ and $\mathbb{Q}[x]$ as additive abelian groups (*i.e.* $\mathbb{Z}$-modules). Prove that

$$\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}[x] \cong \mathbb{Q}[x]$$

as abelian groups. (*Hint.* You may assume that the mapping $r : \mathbb{Q} \times \mathbb{Z}[x] \to \mathbb{Q}[x]$ given by $r(q, f) = qf$ is a surjective balanced map and that every element of $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}[x]$ can be written in the form $q \otimes f$ for some $q \in \mathbb{Q}$ and $f \in \mathbb{Z}[x]$.)

9. Give an example of a UFD that is not a PID and explain why your example works.

10. If R is commutative, sketch a proof of the fact that $r \in R$ is nilpotent if and only if r is contained in every prime ideal of R.

11. Prove that if every R-module is injective, then every R-module is projective

12. If $f : A \to A$ is an R-module homomorphism with $ff = f$, prove that

$$A = Ker\, f \oplus Im\ f$$

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III. Fields and Galois Theory

Do problems 13 and 14 and any two of the remaining four.

13. Prove that every finite field extension is algebraic.

14. Give an example of an algebraic field extension that is not finite and explain why your example works.

15. Suppose that $K \subseteq F$ is an algebraic field extension and that D is an integral domain with $K \subseteq D \subseteq F$. Prove that D is a field.

16. Sketch a proof of the fact that every finite group G is isomorphic to the Galois group of some finite Galois extension $K \subseteq F$.

17. In each case, give a specific example of a *finite* field extension $K \subseteq F$ that satisfies the given condition. Briefly justify your answers.
    (a) $K \subseteq F$ is a separable extension, but is not Galois.
    (b) F is a splitting field over K for some irreducible polynomial $f \in K[x]$, but $K \subseteq F$ is not Galois.

18. Suppose that p is a prime and that K and F are finite fields with $|K| = p^m$ and $|F| = p^n$ for some positive integers m and n. Show that F has a subfield isomorphic to K if and only if $m \mid n$. (*Hint.* For the harder implication, recall that $\mathbb{Z}_p \subseteq F$ is a finite Galois extension with cyclic Galois group and use the Fundamental Theorem.)