Prelim: Linear Algebra and Matrix Theory

Thomas H. Pate

September 2, 2010

Note: I have substituted $\mathbb{C}^{m\times n}$ for the notation $\mathrm{M}_{m,n}(\mathbb{C})$ that I used in class. Both denote the set of all $m\times n$ complex matrices.

1. Give a careful description of the Jordan canonical form of a nilpotent matrix.

2. Give an example of a pair of 4×4 nilpotent matrices that are not similar. Prove from scratch that your matrices are not similar.

3. Suppose D is an $n\times n$ real symmetric matrix such that $D^3 = I$, where I denotes the $n\times n$ identity matrix. Must it be true that $D = I$? Either provide proof, or disprove via counterexample.

4. Suppose $A \in \mathbb{C}^{n\times n}$ and you know the Jordan canonical form of A. Carefully explain how you could determine the minimal polynomial of A.

5. Give an example of a pair of $n\times n$ real matrices that have the same minimal polynomial but are not similar.

6. Characterize all real symmetric matrices whose minimal and characteristic polynomials are the same.

7. What do you know about $n\times n$ real matrices with positive entries? Give the most significant facts, particularly with respect to eigenvalues, eigenvectors, spectral radius etc. Which results extend to non-negative matrices?

8. Suppose $A\in\mathbb{C}^{n\times n}$, and $p(t)$ is a polynomial such that $p(A)=0$. (a.) Show that any eigenvalue of A is a root of $p(A)$. (b.) What does (a.) tell you about the eigenvalues of a matrix D such that $D^2 = I$? (c.) Prove that the minimal polynomial of A divides $p(t)$.

9. Suppose $A \in \mathbb{C}^{n\times n}$ and A is Hermitian. Let λ be an eigenvalue of A of multiplicity one with associated unit eigenvector x. Regarding x as an $n\times 1$ matrix, let $B = A - \lambda x x^*$. (a.) Show that λ is real. (b.) Show that B is Hermitian. (c.) Show that if U is a unitary matrix such that $U^*AU = \Lambda$, where Λ is a diagonal matrix, then U^*BU is also a diagonal matrix, Λ'. What is the relationship between Λ and Λ'?

10. Suppose $A \in \mathbb{C}^{n\times n}$ and A is normal. (a.) Prove that a number $\lambda \in \mathbb{C}$ is an eigenvalue of A if and only if $\bar{\lambda}$ is an eigenvalue of A^*. (b.) Show that the eigenspace of A associated with eigenvalue λ is the same as the eigenspace of A^* associated with eigenvalue $\bar{\lambda}$. (c.) Prove that if λ and μ are distinct eigenvalues of A then the corresponding eigenspaces are orthogonal with respect to standard inner product on $\mathbb{C}^n$. (d.) Using the above prove that if A is normal, then there exists a unitary matrix U such that U^*AU is a diagonal matrix. (e.) Is the converse true?

11. Suppose A is an $n\times n$ complex matrix. Prove from scratch that there exists a unitary matrix U such that U^*AU is upper triangular. Suppose $\mathcal{F}$ is as subset of $\mathbb{C}^{n\times n}$. Under what conditions is it true that there exists a single unitary matrix U such that U^*AU is upper triangular for all $A \in \mathcal{F}$? For extra credit state and prove a theorem that states necessary and sufficient conditions under which members of $\mathcal{F}$ are simultaneously unitarily upper triangularizable.

12. Carefully state the Cauchy interlacing theorem for Hermitian matrices. Suppose $A \in \mathbb{C}^{n\times n}$ and A is Hermitian. If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is the sequence of eigenvalues of A arranged in non-ascending order, and $a = (a_1, a_2, \ldots, a_n)$ is the sequence of diagonal entries of A arranged in non-ascending order, then prove that $\lambda \succeq a$, where $\succeq$ denotes majorization.

13. Suppose $D \in \mathbb{R}^{n\times n}$, and $D = [d_{ij}]$ has non-negative entries. (a.) Show that if each row of D sums to r, where r is a positive real number, then the spectral radius of D is r. (b.) Suspend the assumption that the rows of D sum to the same number and show that the spectral radius of D, $\rho(D)$, is not less than the minimum row sum. That is, show that

$$\rho(D) \;\geq\; \min_{1\leq i\leq n} \left\{ \sum_{j=1}^{n} d_{ij} \right\}.$$

Hint: Let r_i denote the i-th row sum of D and consider the matrix $\tilde{D} = \lambda D$ where λ is the diagonal matrix whose diagonal entries are the reciprocals of the numbers r_i. The case where some $r_i = 0$ must be considered separately. What do you know about the spectral radius of $\tilde{D}$. Let r denote the minimum of the r_i, and let $\hat{D}$ denote $(1/r)D$. What is the relationship between $\tilde{D}$ and $\hat{D}$?

14. Suppose V is a complex vector space of dimention n, and T is a linear map from V to V. Recall that if λ is an eigenvalue of T, then the asssociated generalized eigenspace $GS_\lambda(T)$ is $\ker(T - \lambda I)^n$. (a.) Show that if λ is an eigenvalue of T, then $GS_\lambda(T)$ is invariant under T. (b.) Suppose $\lambda_1, \lambda_2, \ldots, \lambda_k$ is a sequence of distinct eigenvalues of T. Prove directly (from scratch) that if $x_i \in GS_{\lambda_i}(T)$ for each integer i, $1 \leq i \leq k$, and $x_1 + x_2 + \cdots + x_k = 0$, then $x_i = 0$ for each i. (c.) Deduce from this that the sum $+_{i=1}^{k} GS_{\lambda_i}(T)$ is a direct sum. (d.) For extra credit show that if $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ is a complete list of the distinct eigenvalues of T, then $V = \oplus_{i=1}^{k} GS_{\lambda_i}(T)$. (e.) How is the result in (d.) related to the Jordan canonical form?

15. Suppose $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times m}$. (a.) Suppose $\lambda \in \mathbb{C}$, and $\lambda \neq 0$. Show that λ is a eigenvalue of AB, if and only if λ is an eigenvalue of BA. (b.) Assume that λ is a non-zero eigenvalue of AB. Show that the generalized eigenspace $GS_\lambda(AB)$ has the same dimension as the generalized eigenspace $GS_\lambda(BA)$. Hint: Show that $(AB - \lambda I)^k A = A(BA - \lambda I)^k$ for all positive integers k. (c.) Assume that $m \geq n$. Show that $P_{AB}(t) = t^{m-n} P_{BA}(t)$, where $P_{AB}(t)$ and $P_{BA}(t)$ are the characteristic polynomials of AB and BA, respectively.