Algebra Preliminary Exam 2024

Name:__________________________

Group Theory	1	2	3	4	5	Total
Linear Groups & Representations	6	7	8	9	10	
Rings & Modules	11	12	13	14	15	

Instruction:

- Answer each question starting on a new piece of paper, and clearly label each page with which question you're solving.
- Write clearly and legibly.
- Be sure to fully explain all your answers, and give a structured, understandable argument.
- You may quote major results from the textbooks, class notes, and homework, unless you are requested to prove the result by yourself.
- 4 out of 5 questions will be graded for each of the three portions. Please cross out the labels of questions you want to drop in the test.

Group Theory

1. (a) (6 points) Write the permutation as a product of disjoint cycles: $(1\,3\,5)(2\,4\,1)(3\,5\,2)(4\,5\,1)$.

   $(1\ 3)(2\ 5)$

   (b) (6 points) Let $n \geq 5$. Explain why $(1\ 3\ 5)(2\ 4)$ is conjugate to $(1\ 2\ 3)(4\ 5)$ in S_n.

   (c) (8 points) Prove the theorem: two elements of S_n are conjugate if and only if they have the same cycle lengths when expressed as disjoint cycles.

2. In the group $\mathrm{GL}_n(\mathbb{C})$, consider the following subgroups:

$$
\begin{aligned}
G &= \left\{ \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} : \ A \in \mathrm{GL}_r(\mathbb{C}), B \in M_{r,n-r}(\mathbb{C}), C \in \mathrm{GL}_{n-r}(\mathbb{C}) \right\}, \\
D &= \left\{ \begin{bmatrix} A & 0 \\ 0 & C \end{bmatrix} : \ A \in \mathrm{GL}_r(\mathbb{C}), C \in \mathrm{GL}_{n-r}(\mathbb{C}) \right\}, \\
T &= \left\{ \begin{bmatrix} I_r & B \\ 0 & I_{n-r} \end{bmatrix} : B \in M_{r,n-r}(\mathbb{C}) \right\}.
\end{aligned}
$$

(a) (10 points) Prove that G is a semi-direct product of D and T, that is, $G = D \ltimes T = T \rtimes D$ where T is a normal subgroup of G (denoted by $T \trianglelefteq G$).

(b) (10 points) State and prove a sufficient and necessary condition when two elements $\begin{bmatrix} I_r & X \\ 0 & I_{n-r} \end{bmatrix}$ and $\begin{bmatrix} I_r & Y \\ 0 & I_{n-r} \end{bmatrix}$ are conjugate in G.

3. For a subset $A \subseteq \{1, 2, \ldots, n\}$ with $|A| = k$, the action of $\sigma \in S_n$ on A is defined by $\sigma(A) = \{\sigma(a) \mid a \in A\}$. The stabilizer $\mathrm{Stab}_{S_n}(A)$ consists of all permutations in S_n that permute the elements of A among themselves. Let G be a subgroup of S_n.
   (a) (7 points) Show that $\mathrm{Stab}_{S_n}(A)$ is isomorphic to the direct product $S_k \times S_{n-k}$.
   (b) (6 points) Prove that $\mathrm{Stab}_G(A) = G \cap \mathrm{Stab}_{S_n}(A)$.
   (c) (7 points) If G acts transitively on the set of all k-subsets of $\{1, 2, \ldots, n\}$, prove that the only subgroup of S_n containing both G and $\mathrm{Stab}_{S_n}(A)$ is S_n itself.

4. (a) (6 points) State three Sylow theorems for finite groups. No proof is required.
   (b) (7 points) Prove that a group of order p^2, where p is a prime, is abelian.
   (c) (7 points) Determine all groups of order 175 up to isomorphisms.

5. Let (λ, v) be an eigenpair of $A \in M_n(\mathbb{C})$.
   (a) (10 points) If v is the first column of $P \in \mathrm{GL}_n(\mathbb{C})$, prove that $P^{-1}AP = \begin{bmatrix} \lambda & x \\ 0 & A' \end{bmatrix}$ for certain $x \in M_{1,n-1}(\mathbb{C})$ and $A' \in M_{n-1}(\mathbb{C})$.

   (b) (10 points) Prove the Schur Form: for every $A \in M_n(\mathbb{C})$, there exists a unitary matrix $U \in \mathrm{U}_n$ an an upper triangular matrix $B \in M_n(\mathbb{C})$ such that
$$A = UBU^* \tag{1}$$

Linear Groups & Representations

6. Consider the linear groups U_n and SU_n.
   (a) (15 points) Prove that
      - U_n is connected and compact;
      - SU_n is a connected normal subgroup of U_n;
      - $\mathrm{SU}_n \trianglelefteq \mathrm{U}_n$ and $\mathrm{U}_n \,/\, \mathrm{SU}_n \simeq \mathbb{C}^{\times}$.

   (b) (5 points) Let $g \in \mathrm{U}_n$. If $h \in \mathrm{GL}_n(\mathbb{C})$ satisfies that $h^k = g$ for a positive integer k, prove that $h \in \mathrm{U}_n$.

7. (20 points) Explain how you obtain the Lie algebra $Lie(G)$ of a linear Lie group G. Then prove the following:

$$\begin{aligned} Lie(\mathrm{SL}_n(\mathbb{R})) &= \{\text{real } n \times n \text{ trace-zero matrices}\}, \\ Lie(\mathrm{U}_n) &= \{\text{complex } n \times n \text{ skew-Hermitian matrices}\}. \end{aligned}$$

8. Let $T \in \mathrm{GL}_n(\mathbb{C})$. Define the form $\langle , \rangle : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ by

$$\langle v, w \rangle = v^* T w, \quad \text{for } v, w \in \mathbb{C}^n.$$

   (a) (10 points) Prove that the following set of linear operators preserving $\langle , \rangle$ form a linear group:

$$G = \{g \in \mathrm{GL}_n(\mathbb{C}) : \langle gv, gw \rangle = \langle v, w \rangle\} = \{g \in \mathrm{GL}_n(\mathbb{C}) : g^* T g = T\}.$$

   (b) (10 points) Prove that the Lie algebra of G is the following set:

$$Lie(G) = \{X \in M_n(\mathbb{C}) : \langle Av, w \rangle + \langle v, Aw \rangle = 0\} = \{X \in M_n(\mathbb{C}) : A^* T + TA = 0\}.$$

9. (a) (10 points) State the results of irreducible characters of a finite group in terms of number of irreducible characters, orthogonality, and dimensions.

   1. The irreducible characters of G form an orthonormal basis on the space of class functions.
   2. The number r of isomorphism classes of irreducible representations of G equals the number of conjugacy classes of G.
   3. Let $d_i = \dim \chi_i = \dim \rho_i$. Then each $d_i \mid |G|$ and $|G| = d_1^2 + \cdots + d_r^2$.

   (b) (10 points) Find the missing rows in the character table below:

	(1)	(3)	(6)	(6)	(8)
χ_1	1	1	1	1	1
χ_2	1	1	-1	-1	1
χ_3	3	-1	1	-1	0
χ_4	3	-1	-1	1	0
χ_5	2	2	0	0	-1

10. (a) (10 points) Find the character table of the dihedral group

$$D_4 = \langle x,\ y \mid x^4 = y^2 = e,\ xyxy = e \rangle.$$

   (b) (10 points) View D_4 as the symmetry group of a square centered at the origin of $\mathbb{R}^2$ and with vertices e_1, e_2, e_3, and e_4 in order. Let $\rho : D_4 \to \mathrm{GL}_2(\mathbb{C})$ be the representation of D_4 such that $\rho_g(e_i) = e_{g(i)}$ for $g \in D_4$ and $i = 1, 2, 3, 4$. Prove that ρ is an irreducible representation.

D_4	(1) e	(2) (1234)	(2) (13)	(2) (12)(34)	(1) (13)(24)
χ_1	1	1	1	1	1
χ_2	1	-1	-1	1	1
χ_3	1	-1	1	-1	1
χ_4	1	1	-1	-1	1
χ_5	2	0	0	0	-2

Rings & Modules

11. (a) (10 points) Determine the kernel and image of the ring homomorphism $\phi : \mathbb{R}[x] \to \mathbb{C}$ defined by $f(x) \mapsto f(1 - \mathbf{i})$. Show that Im$\phi$ is a 2-dimensional real vector space.

    (b) (10 points) Determine the automorphisms ψ of the polynomial ring $\mathbb{C}[x]$ in which $\psi(z) = \overline{z}$ for $z \in \mathbb{C}$.

12. (a) (10 points) Describe the prime ideals and the maximal ideals in a principal ideal domain.

    (b) (10 points) What are the maximal ideals in the quotient ring $\mathbb{R}[x]/(x^4 - 1)$?

13. (a) (10 points) Prove that the variety of zeros of a set $\{f_1, \dots, f_r\}$ of polynomials depends only on the ideal that they generate.

    (b) (10 points) Which ideals in the polynomial ring $\mathbb{C}[x, y]$ contain $x^2 + y^2 - 29$ and $xy - 10$?

14. (20 points) Define $\varphi : \mathbb{C}[x, y] \to \mathbb{C}[x] \times \mathbb{C}[y] \times \mathbb{C}[t]$ by $f(x, y) \mapsto (f(x, 0), f(0, y), f(t, t))$. Determine the image of this map, and find generators for the kernel.

15. Let R be a commutative ring with identity.

    (a) (10 points) Under what circumstances is an ideal I a free R-module?

    (b) (10 points) Let M be an R-module, and N an R-submodule of M. Prove or disprove: if both N and M/N are free R-modules, then M is a free R-module.