# Prelim Exam- Coding Theory -2005

Do as many of the following as possible (favoring complete answers to fewer parts of questions rather than partial answers to more). The index tables for $\mathbf{F}_4$ and $\mathbf{F}_{16}$ are:

| $i$ | $\alpha^i$ |
|---|---|
| $3=0$ | 10 |
| 1 | 01 |
| 2 | 11 |

| $i$ | $\gamma^i$ |
|---|---|
| $15=0$ | 1000 |
| 1 | 0100 |
| 2 | 0010 |
| 3 | 0001 |
| 4 | 1100 |
| 5 | 0110 |
| 6 | 0011 |
| 7 | 1101 |
| 8 | 1010 |
| 9 | 0101 |
| 10 | 1110 |
| 11 | 0111 |
| 12 | 1111 |
| 13 | 1011 |
| 14 | 1001 |

1. (a) State and prove the Singleton (upper) bound on the size of a q-ary linear code of wordlength n and minimum distance d.

   (b) State and prove the Varshamov (lower) bound on the size of a q-ary linear code of wordlength n and minimum distance d.

2. (a) Give a generator $g(x)$ for a Reed-Solomon code $C$ over $\mathbf{F}_q$ with designed minimum distance $\delta$. What are the parameters of $C$?

   (b) If $q = p^m$, $p$ a prime, what can be said of the parameters of the subfield subcode $C_p$ over $\mathbf{F}_p$? Find a generator polynomial in the case that $q = 16 = 2^4$ and $g(x) = (x+\gamma)\cdots(x+\gamma^6)$.

3. (a) Lift the factorization of $x^{15} - 1 \in \mathbf{Z}_2[x]$ to a factorization of $x^{15} - 1 \in \mathbf{Z}_4[x]$.

   (b) Lift the index table for $\mathbf{F}_{16}$ to $\mathbf{Z}_4$ as well.

4. (a) Prove Forney's formula.

   (b) Use Forney's formula to compute the error magnitude at position $\gamma^4$, given the syndromes $s_1 = \gamma^3$, $s_2 = \gamma^4$, and $s_3 = \gamma^9$ and the error-locator $\sigma(x) = x^3 + \gamma^9 x^2 + \gamma^{13} x + \gamma^{12}$.

5. (a) Which of the following generator matrices define catastrophic convolutional codes:

$$G_1 := \begin{pmatrix} 1+x^2 & 1+x^3 & 1+x+x^3 \end{pmatrix}$$

$$G_2 := \begin{pmatrix} 1+x & 1+x & 1+x^2 \\ x & 1 & x^2 \end{pmatrix}$$

   (b) For each, if catastrophic find a message of infinite weight that produces a finite weight code; and if not, produce a finite state table, and find the minimum distance.

6. Use the Berlekamp-Massey algorithm to find a smallest linear recurrence from the syndromes $s_1 := 0$, $s_2 := \gamma^2$, $s_3 := 0$, $s_4 := \gamma^9$, $s_5 := \gamma^{12}$, $s_6 := \gamma^3$. (As a check, the result should be a scalar multiple of $(x+\gamma)(x+\gamma^2)(x+\gamma^{14})$.)

7. Let $C$ be the AG code defined by $x^2 y + y^2 + x = 0$ in characteristic 2 (similar to the Klein quartic).

   (a) Find those points of $C$ rational over $\mathbf{F}_4$.

   (b) Find the divisors $(x)$ and $(y)$.

   (c) Find local parameters at all points.