Graph Theory Prelim, 7/10/09

1) a) State Kruskal's algorithm for finding a cheapest spanning tree in a connected graph with edge costs.

b) If $\{e_1, e_2, \dots, e_{n-1}\}$ is the edge set of a spanning tree T in the n-vertex graph G, describe a cost assignment to the edges of G so that Kruskal's algorithm is forced to choose the edges of T, in the order listed.

c) What does Kruskal's algorithm choose if it's given a disconnected graph with edge costs?

2) a) State Euler's theorem characterizing those graphs having Euler tours.

b) Let G be a graph having an Euler tour, let e_1 and e_2 be edges of G having exactly one incident vertex in common. Prove or disprove: G has an Euler tour in which e_1 and e_2 are consecutive.

c) Prove (possibly using a)) that any graph G has an orientation in which, at every vertex v of G, the indegree of v and the outdegree of v differ by at most 1.

3) a) Suppose the edge e of the connected graph G is contained in at least one cycle of G. Prove that G\e is connected.

b) Suppose the edge e of the connected graph G is contained in no cycles of G. Prove that G\e has exactly two components. (In this case, e is defined to be a *bridge* of G.)

c) In the case b) above, prove that either every perfect matching of G contains e, or no perfect matching of G contains e.

4) The *Wiener index* of a connected graph G, *W(G)*, is defined to be the sum, over all two-element subsets {v, w} of the vertex set of G, of d(v, w) (= the distance between v and w).

a) Among all connected graphs G on n vertices, prove that the ones with the largest value of W(G) are trees. (Hint: if T is a spanning tree of G, prove that $W(G) \leq W(T)$, with equality if and only if T = G.)

b) Prove that among all trees on n vertices, the path P_n has the largest value of W(G).

c) Among the trees on n vertices, which has the smallest value of W(G)? (Prove your answer.)