Graph Theory Prelim, 2010

1. (a) State Vizing's theorem on the chromatic index (edge-coloring number) of simple graphs.

   (b) Find a graph with chromatic index one billion more than its maximum degree.

   (c) Describe an efficient algorithm for finding the chromatic index of a bipartite graph.

   (d) Describe an efficient algorithm for finding the chromatic index of a tree.

2. (a) Describe an efficient algorithm for finding a cheapest spanning tree in a connected graph with edge costs.

   (b) Since there is no spanning tree of any cost in a disconnected graph, how could you modify this algorithm to find some structure similar to a cheapest spanning tree when applied to a disconnected graph with edge costs?

3. Let m and n be positive integers with $2m < n$. Let G be a simple graph on vertex set $V := \{v_1, v_2, \ldots, v_n\}$ having the following degrees. For $1 \leq i \leq m$, $degree(v_i) = n-1$; for $m+1 \leq i \leq n-m$, $degree(v_i) = n-m-1$; and for $n-m+1 \leq i \leq n$, $degree(v_i) = m$.

   (a) Prove there is only one such graph G, and describe it.

   (b) Show that G contains a cycle of length $n-1$, but none of length n.

4. Let k be a positive integer, and let $X = \{0, 1, 2, \ldots, 2k-1, 2k\}$. Define the simple graph O_k with vertices of O_k all the subsets of X of size k, and two vertices adjacent iff they have no elements in common.

   (a) Draw O_1 and O_2.

   (b) Find the number of vertices of O_k, the degree of each vertex, and the number of edges.

   (c) Prove that O_k contains a cycle of length $2k+1$.

5. State and prove some graph-theoretic corollary to Hall's theorem on systems of distinct representatives. (If you use Hall's theorem to prove it, at least state Hall's theorem.)

6. Show there can be no distance-regular graph with parameters given by

$$B := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 4 & 0 & 2 & 0 \\ 0 & 3 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{pmatrix}.$$

7. Find the chromatic polynomial of the simple graph with adjacency matrix

$$A := \begin{pmatrix} 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Use it to find the number of (vertex) 3-colorings the graph has.

8. Expain by definitions and examples the relationship between regular, distance-regular, distance-transitive, and symmetric graphs.