<u>Graph Theory Prelim – 5/26/12</u>

For this prelim, <u>all</u> graphs are simple, i.e., they have no loops or multiple edges.

1) Let $G = (V, E)$ be a graph, let $u, v \in V$, and let $n$ be a non-negative integer.

   a) Define the following:

      i) u-v walk of length n in G,
      ii) u-v trail of length n in G,
      iii) u-v path of length n in G.

   b) Prove that every path is a trail

   c) Prove that if there is a u-v walk in G, then there is a u-v path in G.

2) An edge e of a graph G is defined to be a *cut edge* if $G\setminus\{e\}$ has more components than G.

   a) Prove that e is a cut edge of G if and only if no cycle of G contains e.

   b) Let $G = (V, E)$ be a graph with $|V| = |E| + 1$. Prove the following are equivalent:

      i) G is connected.
      ii) G has no cycles.
      iii) Every edge of G is a cut edge.
      iv) G is a tree.

   c) Let $T_1$ and $T_2$ be spanning trees of the connected graph G, let e be an edge of $T_1$.

      i) Prove that $(T_1 \setminus\{e\}) \cup \{f\}$ is a spanning tree for some edge f of $T_2$.
      ii) Prove that $(T_2 \setminus\{g\}) \cup \{e\}$ is a spanning tree for some edge g of $T_2$.

(You may assume the following without proof: A tree on n vertices has exactly n-1 edges, and $G\setminus\{e\}$ has at most one more component than G.)

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3) The *chromatic number* of the graph G, $\chi(G)$, is the fewest number of colors required in a proper coloring of the vertices of G. And the *clique number* of G, $\omega(G)$, is the size of a largest set of mutually adjacent vertices of G. Finally, the *complement* of the graph G, $\overline{G}$, is the graph on the same vertex set as G, in which two different vertices are adjacent if and only if they are not adjacent in G. Let C be a cycle on n vertices.

a) If n is odd, prove that in any proper coloring of the vertices of C with colors red, blue, and green, there is at least one green vertex having one red neighbor and one blue neighbor.

b) Prove that $\chi(\overline{C}) = \omega(\overline{C}) + x$, where $x \in \{0, 1\}$, $x \equiv n \pmod 2$, provided that $n \geq 4$.

4) The *chromatic index* of the graph G, $\chi'(G)$, is the fewest number of colors required in a proper coloring of the edges of G. G is *class I* if $\chi'(G) = \Delta(G)$, and *class II* if $\chi'(G) = \Delta(G) + 1$. Vizing's theorem states that every graph is either class I or class II. If x is a real number, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x. (Recall that $\Delta(G)$ is the maximum degree of a vertex of G.)
Define the graph $G = (V, E)$ to be *overfull* if $|E| > \Delta(G)\lfloor |V|/2 \rfloor$.

a) Prove that an overfull graph has an odd number of vertices, and is class II.

b) Let n be an odd integer at least 3, and let G be a graph obtained from the complete graph $K_n$ by removing at most $(n-3)/2$ of its edges. Prove that G is class II.

5) A *matching* in the graph G is a subset of the edges of G, no two of which share a vertex. The matching M in G is *maximal* if M is the only matching in G containing M, and it is *maximum* if there is no matching in G with more edges than M. The *matching number* of G, $\alpha'(G)$, is the size of a maximum matching.

a) Prove that if M is a maximal matching in G, then $|M| \leq \alpha'(G) \leq 2|M|$.

b) For each positive integer k, find a graph G with $\alpha'(G) = 2k$, and having a maximal matching M with $|M| = k$.