ALGEBRA PRELIMINARY EXAM 2018

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Instructions:

- Answer each question starting on a new piece of paper, and clearly label each page with which question you're solving. **Only write on one side of the paper - your work will be scanned electronically for remote grading.**
- Write clearly and legibly.
- Be sure to fully explain all your answers, and give a structured, understandable argument.
- Answers will be graded on clarity and the correctness of the main steps of the reasoning.
- Though much effort has been made to eliminate typos and simple mistakes, if you notice one, ask the proctor. Do not interpret a problem in a way that would make it trivial.
- You may quote major results from the textbook (Lang), class notes, and homework.

Good luck!

Exercise 1. (1) Classify all groups of order 2018.
(2) How many abelian groups of order 2^6 are there?
(3) Classify all groups of order 2^3. Can you make a conjecture for all groups of order p^3?

Exercise 2. Give a example of a polynomial $f(x) \in \mathbb{Q}[x]$ that is not solvable in radicals. Justify your example.

Exercise 3. Consider the group $G = \mathrm{SL}(2, \mathbb{Z}_3)$, the group of 2×2 matrices with entries in the field $\mathbb{Z}_3$ and determinant 1. Let $U < G$ denote the set of unipotent elements in G.

(1) How many elements are in this group?
(2) Is this group cyclic? Give a minimal sized generating set.
(3) Is this group solvable?
(4) Does this group have a subgroup of each order dividing the order of the group?

Note: this question was thrown out. It should have stated that U is the set of upper triangular elements in G.

Exercise 4. Again let $G = \mathrm{SL}(2, \mathbb{Z}_3)$. Consider the action of G on the cartesian product $V = \mathbb{Z}_3 \times \mathbb{Z}_3$ given by matrix-vector product:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix},$$

for $a, b, c, d, x, y \in \mathbb{Z}_3$, and usual arithmetic modulo 3.

(1) Prove that
$$\begin{array}{ccc} G \times V & \to & V \\ (M, v) & \mapsto & Mv \end{array}$$
is a group action.
(2) Compute all orbits of G acting on V.
(3) For one representative in each orbit, compute its stabilizer subgroup.

Exercise 5. Suppose G is a finite group, and A and B are irreducible G-modules. Prove Schur's lemma: *Suppose $\phi \colon A \to B$ is a G-homomorphism. Either A and B are isomorphic, in which case ϕ is a scalar multiple of the identity, otherwise, ϕ is the trivial map.*

Exercise 6. Suppose $0 \to A \to B \to C \to 0$ is an exact sequence of free abelian groups of finite rank.
Prove that $\operatorname{rank}(B) = \operatorname{rank} A + \operatorname{rank} C$.

Exercise 7. Suppose R is a commutative ring with unity. Suppose A and B are R-modules. Recall that the *tensor product* of A and B over R, denoted $A \otimes_R B$ is the R-module generated by all formal symbols $a \otimes b$ (for $a \in A$ and $b \in B$) such that for all $a, a' \in A,\ b, b' \in B,\ r \in R$:

(i) $(a + a') \otimes b = a \otimes b + a' \otimes b$,
(ii) $a \otimes (b + b') = a \otimes b + a \otimes b'$,
(iii) $(ra) \otimes b = a \otimes (rb)$.

Prove the following:

(1) A set A is an abelian group if and only if A is a $\mathbb{Z}$-module.
(2) If A and B are free R-modules, then so is $A \otimes_R B$. What is its rank?
(3) Consider $A = \mathbb{Z}_m$ and $B = \mathbb{Z}_n$ as $\mathbb{Z}$-modules. Identify the $\mathbb{Z}$-module $A \otimes_{\mathbb{Z}} B$.

Exercise 8. (1) If $\alpha, \beta \in \mathbb{C}$ are algebraic numbers, prove that $\alpha + \beta$ is also algebraic.
(2) Given $\deg \alpha = a$ and $\deg \beta = b$, what are the possible degrees of the extension of $\mathbb{Q}$ to $\mathbb{Q}(\alpha + \beta)$?

Exercise 9. Prove that a non-constant polynomial with rational coefficients in one variable of odd degree always has at least one real root.

Exercise 10. Prove two of the following:

- The Fundamental Theorem of Algebra.
- Every principal ideal domain (PID) is a unique factorization domain (UFD).
- Finite groups are reductive.
- Ideals in Noetherian rings (has the ascending chain condition, ACC, for ideals) are finitely generated.