Topology Preliminary Examination, August 25, 2018
Department of Mathematics and Statistics, Auburn University

Instructions:

- Select and solve 8 out of the 15 problems listed on the following page. Only 8 problems can be selected. The remaining problems will not be graded.
- Write your work on the provided paper, leaving one side of each sheet blank. Use different sheets for different problems.
- For each of the selected problems, start your solution on the problem page provided in this package. If necessary, continue your solution on continuation sheets provided separately.
- Fill out the form bellow, and provide the information required on each page of your solution.
- Do not fold any of the exam pages. Put them back in the provided folder: this page on top, then solutions of the problems you selected (in the numerical order, globally and within each problem; pages with solutions up). Do not include not selected problems.

Name: ______________________________

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Circle the selected problems: 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15

Problems for Topology Preliminary Exam, August 25, 2018.

Select and solve 8 out the following 15 problems:

1. Give the definition of a connected space. Prove that the image of a connected space under a continuous map is connected.

2. State the two condition under which a given collection $\mathcal{B}$ of subsets of a set X is a basis for some topology on X.
Define a metric space (X, d).
Define $B_d(x, \epsilon)$ ("open" ϵ-ball centered at x).
Show that $\mathcal{B} = \{B_d(x, \epsilon) \mid x \in X, \epsilon > 0\}$ is a basis for a topology.

3. Give the definition of a Hausdorff space.
Let f and g be two continuous maps of a topological space X into a Hausdorff space Y. Prove that the set $\{x \in X \mid f(x) = g(x)\}$ is closed in X.

4. Give the definition of a compact (topological) space.
Explain what "$\mathbb{R}$ has the least upper bound property" means.
Assuming that $\mathbb{R}$ has the least upper bound property, prove that $[0, 1]$ is compact.

5. Let X and Y be topological spaces. Define the product topology on $X \times Y$.
Prove the following lemma:
Let X and Y be topological spaces with Y compact, and consider the product space $X \times Y$. If $x_0 \in X$ and N is an open set containing $\{x_0\} \times Y$, then there exists a neighborhood W of x_0 in X such that $W \times Y \subset N$.

6. Prove that the product of two compact spaces is compact.

7. Let $f : X \to Y$ be a continuous bijection of a compact space X onto a Hausdorff space Y. Prove that f is a homeomorphism.

8. Let $\{X_\alpha\}_{\alpha \in J}$ be an indexed collection of topological spaces. Define the product and the box topologies on $\prod_{\alpha \in J} X_\alpha$.
Give the definition of a separable space.
Prove that $\mathbb{R}^\omega$ with the the box topology is not separable.

9. Prove that each separable metric space has a countable basis.

10. Define the the lower limit topology on $\mathbb{R}$. Define a normal space.
Let $\mathbb{R}_l$ be the reals with the lower limit topology; prove that $\mathbb{R}_l$ is normal.

11. Give the definition of a complete metric space. Suppose that a metric space (X, d) is complete, and $C_1 \supset C_2 \supset C_3 \supset \ldots$ is a nested sequence of nonempty closed subsets of X such that diam $C_n \to 0$. Prove that the intersection $\bigcap_{n=1}^{\infty} C_n$ is nonempty.

12. Prove that the set of rational numbers is not the intersection of a countable collection of open subsets of $\mathbb{R}$.

13. Suppose that X is a path a connected space, and $x_0, x_1 \in X$. Show there exists an isomorphism of $\pi_1(X, x_0)$ onto $\pi_1(X, x_1)$. (Your isomorphism must be well-defined and all terms used in its definition explained, but you do not need to prove its properties here.)

14. Suppose r is a retraction of a space X onto its subset A. Let $a_0 \in A$. Prove that $r_* : \pi_1(X, a_0) \to \pi_1(A, a_0)$ is a surjection.

15. Let $D = \{x \in \mathbb{R}^2 \mid |x| \leq 1\}$, and let $S^1 = \{x \in \mathbb{R}^n \mid |x| = 1\}$. Outline a proof of the theorem that there is no retraction of the disk D to S^1.