I R A M P
11(2), July 2020, pp. 89-101

ISSN: 2229-39-44

Scalar and Vector Perturbations Addition in the Framework of the Frequency Fluctuation Model

ASTAPENKO V.A[1.], A.YU. LETUNOV[*2;3] AND V.S. LISITSA[1;2;3]

[1]*Moscow Institute of Physics and Technology (National Research University) 141700 Dolgoprudnyj, Moscow Region, Russia*
[2]*National Research Centre \Kurchatov institute", 123182 Moscow, Russia*
[3]*National Research Nuclear University MEPhI, 115409 Moscow, Russia*
[*]*Corresponding Author: E-mail: letunovandrey11@yandex.ru*

ABSTRACT: The problem of the inuence of ion's thermal motions on a spectral line shape in plasmas hasn't been analytically solved yet. The expression for the intensity prole under the action of the Stark broadening is known for two cases. The rst one is applicable for zero ion's temperature (the statistical theory)|the solution to this problem is obtained in the framework of the theory of a vector addition of perturbations from every particle. The second one relates to the limit of high temperature (the impact theory)|this problem has the solution when perturbations from all particles are considered as scalarly additive. The analytical form of a spectral line shape for intermediate values if ion's temperatures is absent. In this work the inuence of ion's thermal motion on the spectral line shape in plasma is considered in terms of the frequency uctuation model (FFM). For the linear Stark eect the FFM gives the wrong dependence of the impact width on the ion's thermal speed. The specic modication which solves this problem is presented in the present paper. The modied FFM consists in the dependence of the jumping frequency on the energy shift. The comparation of the FFM and the GKBO theory for the quadratic Stark eect is presented.

1. INTRODUCTION

The problem of the multiparticle perturbations eect from charged plasma particles on atomic spectra was recognized many years ago [1, 2]. The fundamental solution for the eld strength distribution function from an ensemble of static charged particles was obtained by Holtsmark [3]. The essential property of the Hotsmark function is its dierence from the standard Gaussian distribution. This circumstance is connected with the vector nature of eld strengths sum from individual particles where the contribution of a specic member into the total sum is comparable with the total value. It is in contrast with scalar sum of eld modulus where this contribution is statistically small. The dynamics of Coulomb eld due to particles thermal motion can be recognized from Chandracekhar- Von-Neuman analysis of particles (stars) motion with gravitational interaction [4]. Nevertheless the direct transition from vector to scalar perturbation addition in the Stark broadening theory was not demonstrated analytically up to present time.

The eect of a multiparticle electric eld E on atomic spectra is characterized by the number of particles in the Weiskopf sphere

$$g = N \cdot r_W^3, \tag{1}$$

where N-the concentration of interacting particles and r_W is the Weisskopf radius-the eective particle interaction radius.

The potential of binary interacting particles has the following form

$$V(r) = \frac{C_n}{r^n} \tag{2}$$

where C_n is the interaction constant and r is the distance between particles.

The parameter g can be expressed in terms of C_n and the density of particles. For the linear Stark eect $g = N \cdot \left(\frac{C_2}{v_T}\right)^3$. As for the quadratic Stark effect $g = N \cdot \frac{C_4}{v_T}$. Here v_T is the thermal velocity of the particles. When $g \gg 1$ one can use the statical approximation for line shapes (based on the vector perturbation sum) whereas for the case $g \ll 1$ the impact theory, based on scalar perturbation sum, is valid. Both cases are established in details in the Stark broadening theory (see for example [5]). However, the correct theory for intermediate values of the parameter g is absent. The rst attempt to provide the solution to the problem of arbitrary g for the linear Stark eect was undertaken by V.I. Kogan [6]. He expressed his results for the spectral line prole in terms of complicated path integrals. Unfortunately, he managed to get only the thermal corrections to the statical Holtsmark prole.

A general solution for the quadratic Stark eect was given in the famous paper of H.R. Griem, M. Baranger, A.C. Kolb and G. Oertel (GBKO) [1]. Their result is based on the scalar addition of perturbation following the general scalar theory from older papers on the broadening theory (see [2]). The general GKBO expression for the intensity prole reproduces the impact limit for a high velocities of ions. However when coming to the specic calculations for $g \gg 1$ in helium spectra the authors of [1] used the \vector" static Holtsmark distribution function. Note that the scalar theory is not automatically equal to binary one as it follows from general results for the Van der Waals interaction it (n = 6 in the expression (2)). This problem was considered by Chen and Takeo (CT) in [2].

In 1971 Brissaud and Frisch gave an impetus to the problem of the Stark broadening. They introduced a new approach [7] of the spectral line shape calculation called the Model Microeld Method (MMM). This theory is based on the assumption that alluctuations of electric microeld could be treated as Markovian process. This model has been widely used in the theory of spectral line broadening [8, 9, 10]. Better agreement with the results of modeling the formation of the spectral line shape yielded the Frequency Fluctuation Model (FFM) [11]. The main idea of this theory is that the electric eld uctuations induce uctuations of the radiation intensity and of the radiation frequency. This approach was applied to many problems of spectral line calculations in plasma (see for example [12, 13, 14, 15]). In the work [16] it was shown that the usage of the FFM and the method of the kinetic equation are equivalent. Both of these approaches lead to the same result: the resulting spectral line shape is the functional of the static prole.

Molecular dynamics methods and computer simulations make it possible to perform the most accurate calculations of a spectral line shape in plasmas (see for example [17, 18, 19]). However, with a focus on accuracy, it is necessary to increase the number of particles in the simulation, which requires the use of more computing resources. The FFM is believed to provide the most accurate description of spectral line prole under the action of moving ions and let one perform fast calculation of the intensity prole [20]. However, in the paper [13] it was pointed out that the FFM doesn't reproduce the correct behavior of the prole width in the impact limit for hydrogenic plasma. In the present paper we will show how the FFM can be modied to yield the correct behaviour of the prole in the impact limit. Moreover we will compare the FFM proles with the expression obtained in [1] for the quadratic Stark eect.

The main purpose of this article is to test the FFM procedure. If the FFM describes the inuence of ionic thermal motion on the spectral line shape with a good accuracy it means that the usage of the FFM can replace complicated methods of molecular dynamics and complex analytical solutions like GKBO [1] (which in fact doesn't work for low temperatures). In the present paper we will consider two cases: the linear and quadratic Stark effects.

For the linear Stark eect there is a problem which connected with the incorrect dependence of the impact width on the ion's velocity [13]. In the present paper we will introduce the modication of the FFM procedure which consists in replacing of the constant jumping frequency with the non-constant one obtained in the paper [4]. Our goal is to show that this action leads to the correct behavior of the intensity prole width in the impact limit. In order to do this we will perform analytical and numerical calculations.

The estimations of the impact width in the case of the quadratic Stark effect is one of the goals of the present paper, too. Moreover, we will make the detailed comparison of the FFM with the results of the GKBO theory.

2. A BRIEF DESCRIPTION OF THE FFM

The FFM procedure was suggested in order to overcome the complicated dynamics of a Coulomb ion's microeld [11]. At the rst step the FFM procedure was based on the idea that ion's thermal motion resulting in the eld uctuation would make jumps between dierent Stark components with a frequency, which is equal to

$$\nu = N_i^{1/3} v_{Ti}, \tag{3}$$

where Ni is the density and v_{Ti} is the thermal velocity of ions in plasma.

The corresponding procedure was pure numerical and demonstrated a great success in line shapes description tested by the comparison with computer codes based on molecular dynamics (MMD) [18, 20]. The essential development of the FFM was the theorem about equivalence the FFM procedure to the solution of the kinetic equation with so called \the strong collision integral" describing just the dynamics of ion microeld in terms of jumps between dierent values of the eld strengths responsible for time evolution of Stark components [16]. The presentation of the FFM in terms of the kinetic equation made it possible to obtain analytical solution for the FFM spectral line shapes. The solution contains only the electric eld distribution function (usually Holtsmark) and the jumping frequency so the total FFM shape was the functional of the static eld distribution function. The resulting intensity prole $I(\omega)$ can be expressed as the functional of the normalized statistical prole $W(\omega)$ (see [16])

$$I(\omega) = \frac{1}{\pi} Re \frac{\int \frac{W(\omega')d\omega'}{\nu + i(\omega-\omega')}}{1 - \nu \int \frac{W(\omega')d\omega'}{\nu + i(\omega - \omega')}}, \tag{4}$$

where ω is the energy shift from unperturbed spectral line‡.

The expression (4) can be rewritten in another form that is more convenient for calculations

$$I(\omega) = \frac{\nu}{\pi} \frac{J_0(\omega) J_2(\omega) - J_1^2(\omega))}{J_2^2(\omega) + \nu^2 J_1^2(\omega)}, \tag{5}$$

where

$$J_k(\omega) = \int_{-\infty}^{+\infty} \frac{W(\omega')(\omega - \omega')^k d\omega'}{\nu^2 + (\omega - \omega')^2}. \tag{6}$$

There is the connection between $J_0(\omega)$ and $J_2(\omega)$ which might help one to perform calculations of the spectral line shape faster

$$J_2(\omega) = 1 - \nu^2 J_0(\omega). \tag{7}$$

When $\nu \to 0$ the formula (4) turns into the statistical limit. In the denominator of (4) we can neglect the term which is proportional to ν and obtain the following expression

$$I(\omega) = \frac{1}{\pi} Re \int \frac{W(\omega')(\nu - i(\omega - \omega'))d\omega'}{\nu^2 + (\omega - \omega')^2} = \int W(\omega')\delta(\omega' - \omega) d\omega' = W(\omega) \tag{8}$$

The characteristic value of energy shift of the considered Stark component is equal to

$$\Omega_n^S = C_n^S N_i^{\frac{n}{3}}, \tag{9}$$

where C_n^S is the constant of the Stark eect which depends on the atomic quantum states.

For the linear Stark effect $n = 2$ in the expression (9). In a non-hydrogenic plasma the ionic broadening is determented by the quadratic Stark eect and $n = 4$.

In the present paper we will restrict ourselves to considering a single Stark component. Dierent methods of approximation of the array of radiative transitions are presented in [21, 22, 23]. The considered spectral prole can be treated as a component with and an averaged Stark shift and intensity (see [5]). Note that in the case of the linear Stark eect every spectral component has absolutely the same pair but with negative value of the energy shift (9). It is convenient to use the reduced energy shift which is determined by the following relation

$$z = \frac{\omega}{\Omega_n^S} \tag{10}$$

The expressions (5) and (6) will turn into the following formulas

$$I(z) = \frac{\bar{\nu}}{\pi} \frac{J_0(z)J_2(z) - J_1^2(z))}{J_2^2(z) + \bar{\nu}^2 J_1^2(z)}, \tag{11}$$

$$J_k(z) = \int\limits_{-\infty}^{+\infty} \frac{W(z')(z-z')^k dz'}{\bar{\nu}^2 + (z-z')^2}, \tag{12}$$

where

$$\bar{\nu} = \frac{\nu}{\Omega_n^S}. \tag{13}$$

The relations (7) and (8) remain the same in the framework of the reduced variables.

In the present paper we will use a non-constant jumping frequency. In order to modify the FFM the jumping frequency will be considered as the function of the energy shift $\bar{\nu} \to f(\bar{\nu}, z)$. In this case it is necessary to make the modication of the expressions (11) and (12)

$$I(z) = \frac{1}{\pi} \frac{J_0'(z)J_2(z) - J_1(z)J_1'(z)}{J_2^2(z) + J_1'^2(z)}, \tag{14}$$

where

$$J_k(z) = \int\limits_{-\infty}^{+\infty} \frac{W(z')(z-z')^k dz'}{f^2(\bar{\nu}, z') + (z-z')^2}, \tag{15}$$

$$J_k'(z) = \int\limits_{-\infty}^{+\infty} \frac{f(\bar{\nu}, z')W(z')(z-z')^k dz'}{f^2(\bar{\nu}, z') + (z-z')^2}, \tag{16}$$

It is easy to see that in the case of the constant jumping frequency $f(\bar{\nu}, z) = \bar{\nu}$ the expressions (14)-(16) turn into formulas (11) and (12).

Equations (11)-(16) present a general solution of the spectral line broadening problem describing a smooth transition from static to the impact limit. This method is applicable for arbitrary values of the parameter (1) including $g = 1$.

3. STARK BROADENING OF HYDROGEN SPECTRAL LINES

In this case of the Holtsmark theory the static prole is equal to the following expression ([3, 5])

$$W(z) = \frac{1}{k}H\left(\frac{z}{k}\right), \tag{17}$$

where $k = 2\pi\left(\frac{4}{15}\right)^{\frac{2}{3}} \approx 2.6031$ and H(*x*) is the Holtsmark function which is equal to

$$H(x) = \frac{2}{\pi x}\int_0^\infty t\sin t\exp\left[-\left(\frac{t}{x}\right)^{\frac{3}{2}}\right]dt. \tag{18}$$

In the impact limit, when a velocity of particles is high $(\nu \to \infty)$, the spectral line shape must transform into the Lorentz prole (see for example [5]):

$$L(z) = \frac{1}{\pi}\frac{\gamma}{\gamma^2 + (z - z_0)^2}, \tag{19}$$

where γ is the width of the prole and z0 is the coordinate of the center of the Lorentz prole.

In the case of the linear Stark effect $\gamma \sim \frac{1}{\bar{\nu}}$ [5]. However, in the work [13] the authors showed that the joint usage of the FFM and the quasi-contiguous approximation for the line shape modeling in plasmas [21] leads to the wrong result in the impact limit: $\gamma_{FFM} \sim \frac{1}{\sqrt{\bar{\nu}}}$. We shall show how to reach the correct result for the line width by the application the eld dependent jumping frequency.

In order to give a solution to this problem and overcome this discrepancy we will modify the FFM by introducing of the non-constant jumping frequency $f(\nu, z)$. To perform calculation of the spectral line shape of the single Stark component we will use the expression (14). The eld dependent jumping frequency can be extracted from the paper [4] by interpolation between small and large values of an electric eld strength

$$f_c(\bar{\nu}, z) = \bar{\nu}\frac{3.67}{\tau(z)}, \tag{20}$$

where

$$\tau(z) = \frac{1.37z}{1.37 + z^{\frac{3}{2}}}. \tag{21}$$

While calculating the spectral line shape in a plasma one deals with pairs of the same Stark components, which are symmetric about the center of the spectral line. In the case of a single Stark component integrands in (16) are actually integrated from zero to innity. Consideration of two Stark components implies integration from minus to plus innity. In order to consider this case we need to change the static prole $W(z)$ to $W(|z|)$. The comparsion of these twp case is presented in 1.

We can perform simple analytical estimations of the impact width. In order to do this we will use the fact that all normalized proles $I(z)$, which are symmetric about zero have the following property

$$I(0) \sim \frac{1}{\gamma}, \tag{22}$$

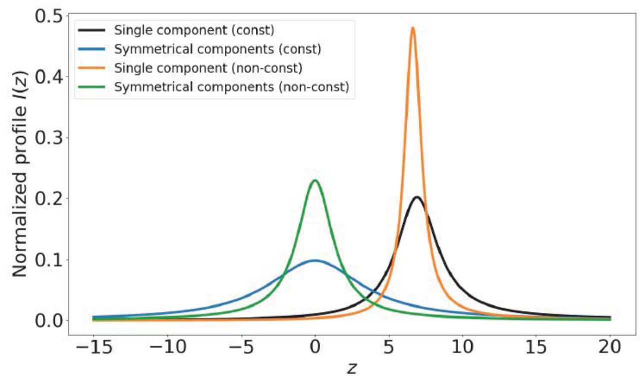

Figure 1. Normalized intensity prole as the function of the reduced energy shift. The calculations were performed for one and two (symmetrical) Stark components. The comparison of two cases of spectral line shape calculation: the usage of constant ("const") and non-constant ("non-const") jumping frequency; $\bar{\nu}$ = 20

where γ is the width of the prole I(z).

Firstly we will obtain the dependence on the impact width in the case of the constant jumping frequency. Using the expression (12) it is easy to see that $J_1(0) = 0$ because we integrate the odd function over the symmetric interval. Using the formula (11) one will obtain

$$I(0) = \frac{\bar{\nu}}{\pi}\frac{J_0(0)}{J_2(0)}. \tag{23}$$

Now we need to estimate the values of $J_0(0)$ and $J_2(0)$ when $\bar{\nu} \gg 1$.

$$J_0(0) = \frac{2}{\bar{\nu}^2}\int\limits_0^{+\infty}\frac{W(z')dz'}{1+\frac{(z-z')^2}{\bar{\nu}^2}} \sim \int\limits_{\bar{\nu}}^{\infty}\frac{W(z')dz'}{1+\frac{(z-z')^2}{\bar{\nu}^2}} \sim \frac{2}{\bar{\nu}^2} + \frac{const_1}{\bar{\nu}^{7/2}} \tag{24}$$

$$J_2(0) = \frac{2}{\bar{\nu}^2}\int\limits_0^{+\infty}\frac{W(z')z'^2dz'}{1+\frac{(z-z')^2}{\bar{\nu}^2}} \sim \frac{const_2}{\bar{\nu}^{3/2}} \tag{25}$$

In these calculations we used the asymptotic behavior of the Holtsmark prole $W(z \gg 1) \sim z^{-5/2}$. While calculating these integrals it should be taken into account that the main contribution to the value of the integral is made by the interval from $\bar{\nu}$ to ∞.

Substitution of (24) and (25) in the expression (23) leads to the following result

$$I(0) \sim \frac{1}{\bar{\nu}^{1/2}}. \tag{26}$$

After comparsion of the expression (26) and (22) we can conclude that for large values of $\bar{\nu}$: $\gamma_{FFM} \sim \frac{1}{\bar{\nu}^{1/2}}$. This dependence of the impact width on the parameter $\bar{\nu}$, obtained for the two symmetrical Stark components, coincides with the result from [13].

In the case of the non-constant jumping frequency we also have to change $f_c(\bar{\nu}, z)$ to $f_c(\bar{\nu}, |z|)$. According to the formula (15)§ the value of $J_1(0)$ is much smaller than all other FFM functions when $z = 0$. Because of that we can neglect the second terms both in nominater and denominater of the expression (14). It leads to the following formula for the non-constant jumping frequency

$$I(0) = \frac{1}{\pi}\frac{J_0'(0)}{J_2(0)}. \tag{27}$$

Using again the asymptotic behavior of the Holtsmark prole and the formula for Chandrasekhar|von Neumann jumping frequency (20) we will estimate the values of J_0 (0) and $J_2(0)$.

$$J_0'(0) \sim \frac{1}{\bar{\nu}}, \tag{28}$$

$$J_2(0) \sim \frac{1}{\bar{\nu}^2}, \tag{29}$$

$$I(0) \sim \frac{1}{\bar{\nu}}. \tag{30}$$

Formulas (27)—(30) leads to the following dependence of the impact width on $\bar{\nu}$: $\gamma_{FFM} \sim \frac{1}{\bar{\nu}}$. According to the basic results of the impact theory [5] this is correct behavior of the prole width.

The normalized intensity proles as the function of the reduced energy shift is presented in the gure 2. Using the expression (14) we performed calculations for the spectral line shape of two symmetrical Stark components in the two cases: $f(\bar{\nu}, z) = \bar{\nu}$ (constant) and $f(\bar{\nu}, z) = f_c(\bar{\nu}, z)$ according to the formula (20) (non-constant). For small ion velocities or high densities, according to the relation (8), the intensity proles are close to each other. As the parameter $\bar{\nu}$ grows, the dierence between graphs increases. For large values of $\bar{\nu}$ width of proles calculated using non-constant jumping frequency is noticeably lower than in the case of constant $f(\bar{\nu}, z) = \bar{\nu}$.

The intensity proles presented in the gure 2 can be approximated by the function (19) with a good accuracy. Comparsion of the FFM proles in the cases of constant and non-constant jumping frequencies with it's approximation by the function (19) is presented in the gure 3. In this calculation we put $\bar{\nu}$ = 15. In order to do this approximation we used the the method of least squares. It is easy to see that the FFM reproduces Lorentzian prole in the impact limit. However, the width of the proles in the cases of the constant jumping frequency $f(\bar{\nu}, z) = \bar{\nu}$ and non-constant $f(\bar{\nu}, z) = f_c(\bar{\nu}, z)$ have completely dierent values. One of the goals of the present paper is to conrm the analytical results (26),(30) for the impact width $\gamma = \gamma_{FFM}$ by numerical calculations. In order to do that we will calculate the FFM proles with dierent values

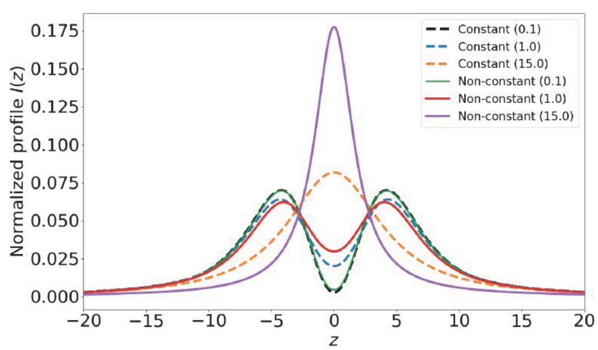

Figure 2. The normalized intensity prole of two symmetrical Stark components as the function of the reduced energy shift. Comparison of the FFM proles calculated with the constant and non-constant jumping frequencies. Calculations are presented for dierent values of ion's velocity. The value of the reduced constant jumping frequency $\bar{\nu}$ is written in the brackets.

of $\bar{\nu}$. Then every intensity prole will be approximated by the Lorentzian (19). After this tting we will approximate the obtained impact width by the following functions

$$y_c = \frac{A}{\sqrt{\bar{\nu}}}, \tag{31}$$

for the constant jumping frequency

$$y_n = \frac{B}{\bar{\nu}}, \tag{32}$$

for the non-constant jumping frequency.

In the gure (4) one can see the values of the impact width as the function of the constant jumping frequency $\bar{\nu}$||. The specic curves (31)-(32) that approximate the function $\gamma_{FFM} = \gamma_{FFM}(\bar{\nu})$ are also presented in the gure 4. We used the method of least squares to approximate the function $\gamma_{FFM}(\bar{\nu})$.

The gure 4 shows that the analytical results for the impact width are in agreement with numerical calculations. Withing the margin of numerical error (double usage of the least squares method) we can conclude that γ_{FFM} can be approximated by the functions (31) and (32) with a good accuracy.

||Situation when $\bar{\nu} \gg 10$ rarely realized for ions in practice. Realistic values of one can nd in [5]. Calculations for such big $\bar{\nu}$ were performed to show how the FFM formally turns into the impact theory.

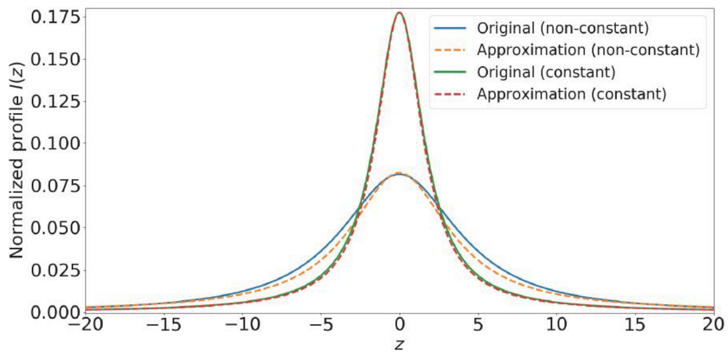

Figure 3. The normalized intensity prole of two symmetrical Stark components as the function of the reduced energy shift. Comparison of the FFM proles calculated with constant and non-constant jumping frequency and it's approximation by the Lorentz prole; $\bar{\nu}$ = 15

To sum all up, the modication of the FFM, which consists in the usage of the non-constant jumping frequency, yield the correct results for the limit of low and hight temperatures. For the case of $\bar{\nu} \sim 1$ the proles calculated with constant and non-constant jumping frequency are close to each other. The results presented in [18, 20] shows that when $\bar{\nu} \sim 1$ the FFM calculations are in agreement with computer modelling. So we can conclude that the modied FFM gives the correct description of the spectral line shape for all temperatures.

4. STARK BROADENING OF NON-HYDROGENIC SPECTRAL LINES

In the case of non-hydrogenic atoms the ionic broadening is determined by the quadratic Stark eect. It leads to the transformation of the static prole *W*(*z*)

$$W_q(z) = \frac{1}{2\sqrt{z}} W(\sqrt{z}), \tag{33}$$

where $W_q^v(z)$ is the statical prole of the single Stark component for the energy shift which is proportional to the squared electric led.

The expression (17) will take form

$$W_q^v(z) = \frac{1}{2\sqrt{z}k} H\left(\frac{\sqrt{z}}{k}\right), \tag{34}$$

where the upper index 'v' means that this prole was obtained using the vector addition theory.

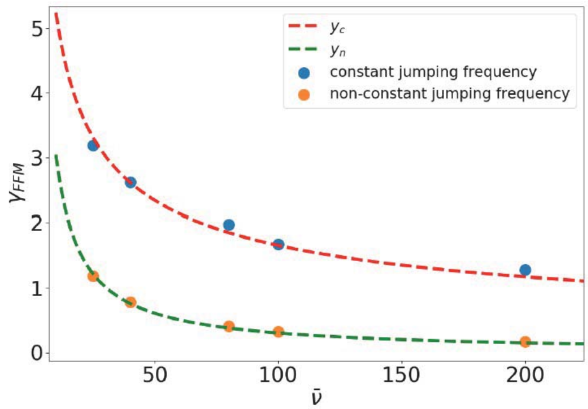

Figure 4. The impact width as the function of the constant jumping frequency $\bar{\nu}$. The comparison of two cases of spectral line shape calculation: the usage of constant and non-constant jumping frequency; Also presented the specic curves $y_c = \frac{16.6}{\sqrt{\bar{\nu}}}$ and $y_n = \frac{30.4}{\bar{\nu}}$ which approximate the dependence of the impact width on $\bar{\nu}$.

Using the theory of the scalar addition of perturbations the authors in works [1] (GKBO) and [2] (CT) obtained the statical prole in the case of the quadratic Stark effect

$$W_q^s(z) = C^s \cdot \int_0^\infty \cos{(zt - at^{\frac{3}{4}})} \exp{(-bt^{\frac{3}{4}})} dt, \tag{35}$$

where a = 14:0309 and b = 5:81178;

$$C^s = \Big[\int_{-\infty}^{+\infty} dz \int_0^\infty \cos{(zt - at^{\frac{3}{4}})} \exp{(-bt^{\frac{3}{4}})} dt \Big]^{-1} \tag{36}$$

It has been already underlined earlier that the scalar addition theory is no applicable in the static case. However it would be interesting to compare the Holtsmark prole (34) with the expression (35).

Figure (5) shows graphs of the functions (34) and (35). For small values of z one can see a big dierence between two proles. The scalar one is equal to zero for $z < 8:1$. This strange result is connected with strong oscillations of the integrand in the formula (35). Both of these functions decrease much slower than the Holtsmark static prole for the linear Stark effect.

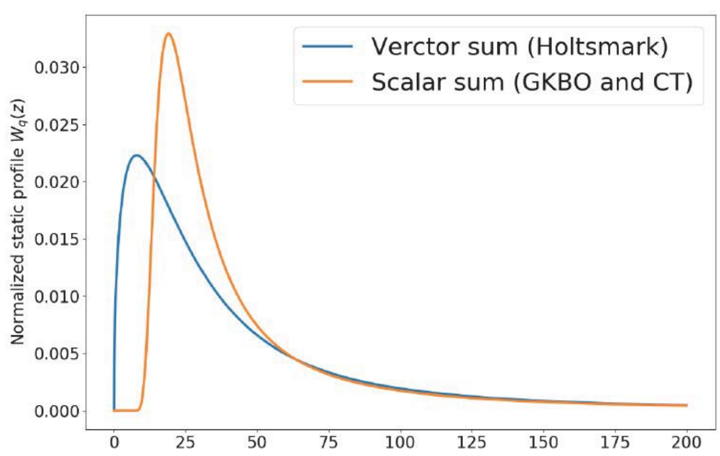

Figure 5. The normalized static prole as the function of the reduced energy shift. The comparison of two cases: vector (Holtsmark) and scalar (CT and GKBO) perturbation addition theories.

In the work [2] the authors obtained the general expression for a static prole for any type of interracion (in the framework of the theory of scalar perturbation addition). However for the linear Stark eect formula for the static prole is expressed in terms of the divergent integrals. That why we don't provide a comparison similar to shown in the gure (5).

Using the theory of scalar addition of perturbations the authors of [1] obtained the intensity prole (GKBO) of a single component for any ion's temperature

$$I_{GKBO}(\bar{\nu}, z) = \frac{1}{\pi} Re\Big[\int_0^\infty dt \exp\left(izt + g(\bar{\nu}, t)\right)\Big], \tag{37}$$

where

$$g(\bar{\nu}, t) = 2\pi\bar{\nu}\int_0^\infty \rho d\rho \int_{-\infty}^{+\infty} dt_1 \Big(\exp\left\{-i\frac{1}{2\bar{\nu}\rho^3}\varphi(\bar{\nu}, t, t_1, \rho)\right\} - 1\Big), \tag{38}$$

$$\varphi(\bar{\nu}, t, t_1, \rho) = \arctan\left\{\frac{\bar{\nu}(t-t_1)}{\rho}\right\} + \arctan\left\{\frac{\bar{\nu}t_1}{\rho}\right\} + \frac{\bar{\nu}(t-t_1)\rho}{\rho^2 + \bar{\nu}^2(t-t_1)^2} + \frac{\bar{\nu}t_1\rho}{\rho^2 + \bar{\nu}^2 t_1^2} \tag{39}$$

For large values of $\bar{\nu}$ the expression (37) reproduces the impact theory for the quadratic Stark effect

$$I_{GKBO}(\bar{\nu} \gg 1, z) = \frac{1}{\pi}\frac{\gamma_{GKBO}}{\gamma_{GKBO}^2 + (z - \Delta_{GKBO})^2}, \tag{40}$$

where

$$\gamma_{GKBO} = \bar{\nu}^{\frac{1}{3}}\pi\left(\frac{\pi}{2}\right)^{\frac{2}{3}}\Gamma\left(\frac{1}{3}\right)\cos\frac{\pi}{3} = 5.6863\bar{\nu}^{\frac{1}{3}}, \tag{41}$$

$$\Delta_{GKBO} = \bar{\nu}^{\frac{1}{3}}\pi\left(\frac{\pi}{2}\right)^{\frac{2}{3}}\Gamma\left(\frac{1}{3}\right)\sin\frac{\pi}{3} = 9.8489\bar{\nu}^{\frac{1}{3}}, \tag{42}$$

where $\Gamma(z)$ is the gamma function.

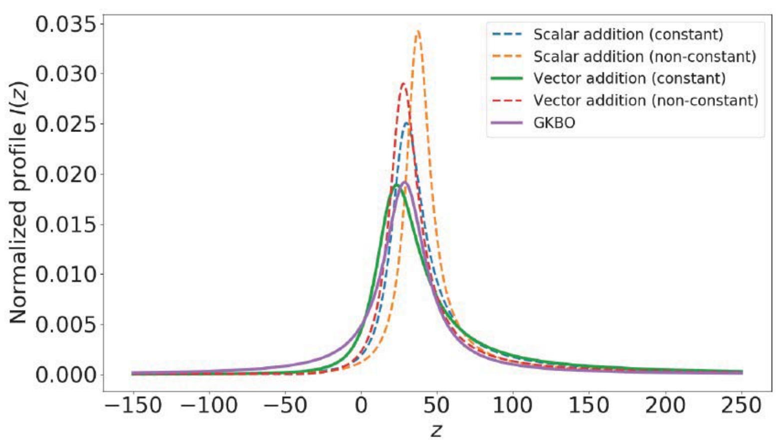

Figure 6. The normalized intensity prole as the function of the reduced energy shift. The comparison of the FFM proles calculated using the theories of scalar and vector addition as well as constant and non-constant jumping frequencies with GKBO proles; $\bar{\nu}$ = 25

The comparison of the FFM and GKBO proles for $\bar{\nu}$ = 25 is presented in the gure (6). These calculations show that the usage of the constant jumping frequency works much better than non-constant. The closest to GKBO graph is the FFM prole with constant frequency and Holtsmark static prole (34).

The comparison of the FFM and GKBO proles is presented in the gure (7). Firstly, the usage of the Holtsmark distribution (34) as the static prole in the FFM procedure gives better results for high temperatures than scalar one (35). Secondly, when $\bar{\nu}$ < 1 the shape of the FFM proles changes very slowly. Finally, the FFM spectral line shapes for the quadratic Stark eect are asymmetric about the center of the prole. As we can see from the gures (6) and (7) for $\bar{\nu} \gg 1$ the joint usage of the FFM and the theory of vector addition of perturbations gives the results which are close to the GKBO proles.

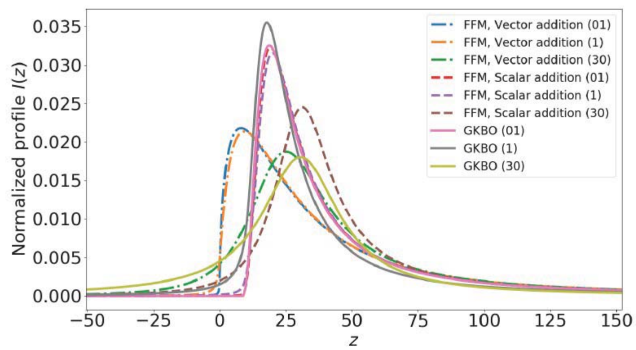

Figure 7. The normalized intensity prole as the function of the reduced energy shift. The comparison of the FFM proles calculated using the theories of scalar and vector addition with GKBO proles; Values of are written in the brackets.

The asymptotic estimations for $\bar{\nu} \gg 1$, similar to those that were done for the Linear Stark eect, of the integrals (13) leads to the following results

$$\left\{ J_0(0) \sim \frac{1}{\bar{\nu}^2}, \qquad J_1(0) \sim \frac{1}{\bar{\nu}^{7/4}}, \qquad J_2(0) \sim \frac{1}{\bar{\nu}^{3/4}}. \right. \tag{43}$$

It is easy to see that in the framework of asymptotics (43) the second terms both in nominator and denominater of (11) could be neglected. Again we can use the relation (22). In this case $\gamma_{FFM} \sim \bar{\nu}^{1/4}$, which is slightly dierent from the correct GKBO result. As numerical calculations show this "impact" asymptotic starts working for $\bar{\nu} \sim 10^4$. This slow convergence to the impact asymptotic is connected with the fact that now the FFM prole center doesn't located at zero. These estimations works when we can neglect the shift of the prole compared to its width. However, for lower $\bar{\nu}$ the coordinate of the maximum of the prole is of the same order of magnitude as it's width. Such values of $\bar{\nu}$ are non-realistic for ions in plasmas. So this asymptotic of the impact width should be treated just as mathematical fact.

Direct use of the formula (37) will make calculations of a spectral line shape very cumbersome. Moreover the GKBO results are incorrect for low temperatures (high densities). The FFM allows one to perform calculations of a spectral line shape much faster and it gives correct results for low temperatures (high densities). The numerical calculations, analogous to those which was done for the linear Stark effect shows that in the impact limit the prole width changes slowly. This means that the FFM produce the impact limit for the quadratic Stark effect ($\gamma \sim \bar{\nu}^{1/3}$). The veried analytical theory for the case of $\bar{\nu} \sim 1$ doesn't exist. However, in the work [20] it was shown that the FFM spectral line shape is close to the prole obtained by the usage of method of molecular dynamics for $\bar{\nu} \sim 1$. We can conclude that the FFM can be a good approximation for the intensity prole of a Stark component when 0 < $\bar{\nu}$ < 30.

5. CONCLUSION

The problem of the inuence of ion's thermal motion on the shape of spectral line in plasmas hasn't been solved yet. There is the strict analytical solution for the case of zero temperature of ions. This result is obtained in the framework of the theory of the vector addition of perturbations from every particle. Also V.I. Kogan obtained the thermal corrections to this result [6]. However if we change vector addition of perturbations to scalar [1]) we will obtain the result which is correct for high temperatures. Unfortunately, the strict theory for intermediate values of temperature hasn't been constructed yet.

Consideration of the ionic motion as a stochastic process leads us to the FFM procedure. The main results of the FFM is that spectral line shape in a plasma with arbitrary temperature depends on the static prole and the jumping frequency (3) (see formula (4)). It allows one to perform fast calculations of the intensity profile.

In the impact limit the FFM procedure with the constant jumping frequency gives the wrong dependence of the impact width on the ion's thermal velocity in the case of the linear Stark eect (see the work [13] and the gure 4). However, if we change the constant jumping frequency to the expression (20), obtained in the work [4], the FFM will reproduce the impact limit. Thus, we can strictly assert that the modied FFM gives the correct results in the limits of low and high temperatures. The numerical analysis and comparsion with computer modeling [20] of an ion's microeld dynamics shows that the FFM works good in the case of $\bar{\nu} \sim 1$ (see formula (13). We can conclude that the modied FFM is applicable for any ionic temperatures and densities.

For the quadratic Stark eect the FFM procedure doesn't turn the static prole to the impact one. Analytical estimations shows that for large values of $\bar{\nu}$ the behavior of the impact width is close to the correct, but doesn't exactly match $(\gamma_{GKBO}/\gamma_{FFM} \sim \bar{\nu}^{1/12})$. Note the convergence of the FFM prole to the impact asymptotic is very slow. It is achieved at non-physical values of $\bar{\nu}$. However for high ionic temperatures the joint usage of the FFM with constant jumping frequency and the Holtsmark theory (vector addition) yields the results which are very close to the GKBO proles. The numerical analysis demonstrated in the present paper and the results from the work [20] shows that the FFM approximates the spectral line shape of a Stark broadened prole in the physical domain $0 < \bar{\nu} < 30$.

References

[1] HR Griem, M Baranger, AC Kolb, and G Oertel. Stark broadening of neutral helium lines in a plasma. Physical Review, 125(1):177, 1962.

[2] M. Takeo and S. Chen. Broadening and shift of spectral lines due to the presence of foreign gases. Reviews of modern physics, 29(1):20, 1957.

[3] Johan Holtsmark. nUber die verbreiterung von spektrallinien. Annalen der Physik, 363(7):577{630, 1919.

[4] S. Chandrasekhar and J von Neumann. The statistics of the gravitational eld arising from a random distribution of stars. Astrophysical Journal, 95:489{531, 1942.

[5] I. I. Sobelman. Introduction to the Theory of Atomic Spectra: International Series of Monographs in Natural Philosophy, volume 40. Elsevier, 2016.

[6] V.I. Kogan. Broadening of spectral lines in high-temperature plasma. Plasma physics and the problem of controlled fusion, 1958.

[7] A Brissaud and U Frisch. Theory of Stark broadening|ii exact line prole with model microeld. Journal of Quantitative Spectroscopy and Radiative Transfer, 11(12):1767{1783, 1971.

[8] Alain Brissaud, C Goldbach, J Leorat, A Mazure, and G Nollez. On the validity of the model microeld method as applied to Stark broadening of neutral lines. Journal of Physics B: Atomic and Molecular Physics, 9(7):1129, 1976.

[9] Billy W Acon, Chantal Stehl–e, Hao Zhang, and Akbar Montaser. Stark-broadened hydrogen line proles predicted by the model microeld method for calculating electron number densities. Spectrochimica Acta Part B: Atomic Spectroscopy, 56(5):527{539, 2001.

[10] C Stehl–e and R Hutcheon. Extensive tabulations of Stark broadened hydrogen line proles. Astronomy and Astrophysics Supplement Series, 140(1):93{97, 1999.

[11] Bernard Talin, Annette Calisti, Laurence Godbert, Roland Stamm, RW Lee, and Lewis Klein. Frequency-uctuation model for line-shape calculations in plasma spectroscopy. Physical Review A, 51(3):1918, 1995.

[12] Sandrine Ferri, Annette Calisti, Caroline Moss–e, Laurence Mouret, Bernard Talin, Marco A Gigosos, Manuel A Gonz–alez, and Valery Lisitsa. Frequency-uctuation model applied to Stark-Zeeman spectral line shapes in plasmas. Physical Review E, 84(2):026407, 2011.

[13] E Stambulchik and Y Maron. Quasicontiguous frequency-uctuation model for calculation of hydrogen and hydrogenlike Stark-broadened line shapes in plasmas. Physical Review E, 87(5):053108, 2013.

[14] Caroline Moss–e, Annette Calisti, Roland Stamm, Bernard Talin, LA Bureyeva, and VS Lisitsa. A universal approach to rydberg spectral line shapes in plasmas. Journal of Physics B: Atomic, Molecular and Optical Physics, 37(6):1343, 2004.

[15] Andrei Letunov and Valery Lisitsa. The Coulomb Symmetry and a Universal Representation of Rydberg Spectral Line Shapes In Magnetized Plasmas. Symmetry, 12(11):1922, 2020.

[16] LA Bureeva, MB Kadomtsev, MG Levashova, VS Lisitsa, A Calisti, B Talin, and F Rosmej. Equivalence of the method of the kinetic equation and the uctuating-frequency method in the theory of the broadening of spectral lines. JETP letters, 90(10):647{650, 2010.

[17] Bernard Talin, Emmanuelle Dufour, Annette Calisti, Marco A Gigosos, Manuel A Gonzalez, Teresa del Rio Gaztelurrutia, and JamesWDufty. Molecular dynamics simulation for modelling plasma spectroscopy. Journal of Physics A: Mathematical and General, 36(22):6049, 2003.

[18] Sandrine Ferri, Annette Calisti, Caroline Moss–e, Jonel Rosato, Bernard Talin, Spiros Alexiou, Marco A Gigosos, Manuel A Gonz–alez, Diego Gonz–alez-Herrero, Natividad Lara, et al. Ion dynamics eect on Stark-broadened line shapes: A cross-comparison of various models. Atoms, 2(3):299{318, 2014.

[19] J. Rosato, Y. Marandet, and R. Stamm. A new table of Balmer line shapes for the diagnostic of magnetic fusion plasmas. Journal of Quantitative Spectroscopy and Radiative Transfer, 187:333{337, 2017.

[20] A. Calisti, Caroline Moss–e, Sandrine Ferri, Bernard Talin, Frank Rosmej, LA Bureyeva, and V.S. Lisitsa. Dynamic Stark broadening as the dicke narrowing eect. Physical Review E, 81(1):016406, 2010.

[21] Evgeny Stambulchik and Yitzhak Maron. Stark eect of high-n hydrogen-like transitions: quasicontiguous approximation. Journal of Physics B: Atomic, Molecular and Optical Physics, 41(9):095703, 2008.

[22] S.A. Gulyaev. Prole of the Hn alfa radio lines in a static ion eld. Soviet Astronomy, 22:572{576, 1976.

[23] A. Yu. Letunov and V.S. Lisitsa. Stark{Zeeman and Blokhintsev spectra of Rydberg atoms. Journal of Experimental and Theoretical Physics, 131(5):696{706, 2020.