I R A M P
11(2), July 2020, pp. 107-113

ISSN: 2229-39-44

Rydberg States of Muonic-Electronic Helium and Heliumlike Ions: The Role of the Eccentricity of the Classical Orbits

N. KRYUKOV[1] AND E. OKS[2,*]

[1]*Universidad Nacional Autónoma de México, Av. Universidad 3000, col. Ciudad Universitaria, del. Coyoacán, México, DF 04510, Mexico*
[2]*Physics Department, 380 Duncan Drive, Auburn University, Auburn, AL 36849, USA*
** Corresponding author email: goks@physics.auburn.edu*

ABSTRACT: There are lots of studies of muonic atoms and molecules because of their various applications. In the previous paper by one of us, there were analyzed Rydberg states of the muonic-electronic helium atom or helium-like ion and there was used the fact that the muon motion occurs much more rapidly than the electron motion. Assuming that the muon and nucleus orbits are circular, he applied the analytical method based on separating rapid and slow subsystems. He showed that the electron moves in an effective potential that is mathematically equivalent to the potential of a satellite orbiting an oblate planet like the Earth. He also showed that the "unperturbed" elliptical orbit of the electron engages in two precessions simultaneously: the precession of the electron orbit in the plane of the orbit and the precession of the orbital plane of the electron around the axis perpendicular to the plane of the muon and nuclear orbits. In the present paper we study how the allowance for a relatively low eccentricity ε of the muon and nucleus orbits affects the motion of the electron. We derive an additional, ε-dependent term in the effective potential for the motion of the electron. We show analytically that in the particular case of the planar geometry (where the electron orbit is in the plane of the muon and nucleus orbits), it leads to an additional contribution to the frequency of the precession of the electron orbit. We demonstrate that this additional, ε-dependent contribution to the precession frequency of the electron orbit can reach the same order of magnitude as the primary, ε-independent contribution to the precession frequency. Therefore, the results of our paper seem to be important not only qualitatively, but also quantitatively.

Keywords: muonic-electronic helium; muonic-electronic helium-like ions; Rydberg states; role of the eccentricity of the muon orbit; precession of the electron orbit

1. INTRODUCTION

There are lots of studies of muonic atoms and molecules because of their various applications – see, e.g., papers [1-8] and references therein. In particular, in paper [6] the author considered Rydberg states of the muonic-electronic helium atom or helium-like ion and used the fact that the muon motion occurs much more rapidly than the electron motion. Therefore, he applied the analytical method centered at separating rapid and slow subsystems. He showed that the electron moves in an effective potential that is mathematically equivalent to the potential of a satellite orbiting an oblate planet (the Earth satellite being an example).

Further, in paper [6] it was shown that the "unperturbed" elliptical orbit of the electron engages in the following two precessions simultaneously: the precession of the electron orbit in the plane of the orbit and the precession of the orbital plane of the electron around the axis of symmetry of the muonic orbit. Despite these two precessions, the elliptical orbit of the Rydberg electron does not change its shape. The fact that the area of the elliptical orbit is conserved manifests the conservation of the square of the electron angular momentum. Thus, the system has higher than geometrical symmetry: indeed, from the geometrical symmetry (which is axial) followed only the conservation of the projection of the angular momentum on the axis of symmetry. This was a counterintuitive result of general physical interest.

In paper [6] the muon and nucleus orbits were considered circular. In the present paper we consider a more general situation where the muon and nucleus orbits are elliptical. We study how the allowance for a relatively low eccentricity ε of the muon and nucleus orbits affects the motion of the electron. We derive an additional, ε-dependent term in the effective potential for the motion of the electron. We show analytically that in the particular case of the planar geometry (where the electron orbit is in the plane of the muon and nucleus orbits), it leads to an additional contribution to the frequency of the precession of the electron orbit. We demonstrate that this additional, ε-dependent contribution to the precession frequency of the electron orbit can reach the same order of magnitude as the primary, ε-independent contribution to the precession frequency.

2. NEW RESULTS

As in paper [6], in this study we analyze the system consisting of a muon, an electron and a nucleus of charge Z. Both leptons are in Rydberg states such that their principal quantum numbers are $n_\mu >> 1$ (for the muon) and $n_e >> 1$(for the electron). The electron is much further way from the nucleus than the muon – due to the large difference in the masses of the two leptons.

In this study we consider the case of low-eccentricity orbits of the muon and the nucleus. For a Coulomb potential

$$U = -\frac{\alpha}{r} \tag{1}$$

(where $\alpha = Ze^2$, e being the electron charge), the equation of the motion in the orbital plane is

$$\frac{p}{r} = 1 + \varepsilon\cos\varphi \tag{2}$$

Here

$$p = \frac{L^2}{m_r\alpha}, \varepsilon = \sqrt{1 + \frac{2EL^2}{m_r\alpha^2}} \tag{3}$$

where ε is the eccentricity, (r, φ) are the polar coordinates, E is the energy, L in the angular momentum, and m_r is the reduced mass of the subsystem “particle – Coulomb center”.

For low-eccentricity orbits (ε << 1), we have

$$r = \frac{p}{1 + \varepsilon\cos\varphi} \approx p(1 - \varepsilon\cos\varphi) = r_0(1 - \varepsilon\cos\varphi) \tag{4}$$

where r_0 is the radius of the circular orbit for ε = 0. As the case ε = 0 was analyzed in paper [6], the Rydberg electron perceives the rapid subsystem (the nucleus and the muon) as two uniformly charged rings of radii R_{nucl0} and $R_{\mu 0}$, where $R_{\mu 0}/R_{nucl0} = m_{nucl}/m_\mu >> 1$ (m_{nucl} is the mass of the nucleus). The effective potential for the Rydberg electron in that case was

$$U_{eff}^{(0)} = -\frac{(Z-1)e^2}{r} - \frac{e^2(R_{\mu 0}^2 - ZR_{nucl0}^2)}{4r^3}(3\cos^2\theta - 1) \tag{5}$$

In our case of ε << 1, the quantities R_μ and R_{nucl} are the following functions of time:

$$R_\mu = R_{\mu 0}(1 - \varepsilon\cos\Omega t), R_{nucl} = R_{nucl0}(1 - \varepsilon\cos\Omega t) \tag{6}$$

where

$$\Omega = \{Ze^2/[m_{\mu r}(R_{\mu 0} + R_{nucl0})^3]\}^{1/2} \approx [Ze^2/(m_{\mu r}R_{\mu 0}^3)]^{1/2} \tag{7}$$

is the frequency of revolution of the muon and of the nucleus about their center of mass, and

$$m_{\mu r} = \frac{m_\mu m_{nucl}}{m_\mu + m_{nucl}} \tag{8}$$

is the reduced mass of the pair "nucleus – muon". Substituting Eq. (6) into Eq. (5), we obtain the following time-dependent "potential":

$$U(t) = U_{eff}^{(0)} + \frac{e^2(R_{\mu 0}^2 - ZR_{nucl0}^2)}{2r^3}(3\cos^2\theta - 1)\varepsilon\cos\Omega t \equiv U_{eff}^{(0)} + W(r, \cos\theta)\cos\Omega t \tag{9}$$

The term $W(r, \cos\theta)\cos\Omega t$ can now be processed by the method of effective potentials [9-13] with respect to the totally unperturbed Hamiltonian

$$H_0 = \frac{p^2}{2m_{er}} - \frac{(Z-1)e^2}{r} \tag{10}$$

where

$$m_{er} = \frac{m_e(m_\mu + m_{nucl})}{m_e + m_\mu + m_{nucl}} \tag{11}$$

is the reduced mass of the electron orbiting the nucleus-muon pair. For helium ($Z = 2$), $m_{er} = 0.9999$, so that for any Z, m_{er} is very close to unity. Also, because $R_{\mu 0}/R_{nucl0} >> 1$, R_{nucl0} can be ignored in the potential.

The zeroth-order effective potential,

$$U_0 = \frac{1}{4\Omega^2}[W, [W, H_0]] = \frac{9\varepsilon^2 e^4 R_{\mu 0}^4(1 - 2\cos^2\theta + 5\cos^4\theta)}{16 m_{er}\Omega^2 r^8} \tag{12}$$

where W is defined in Eq. (9) and $[P, Q]$ are the Poisson brackets, is the time-independent term for the effective potential. On substituting Ω from Eq. (7) in Eq. (12), we get:

$$U_0 = \frac{9\varepsilon^2 m_{\mu r} e^2 R_{\mu 0}^4(1 - 2\cos^2\theta + 5\cos^4\theta)}{16 m_{er} Z r^8} \tag{13}$$

Therefore, the complete effective potential in this case is

$$\begin{aligned} U_{eff} &= U_{eff}^{(0)} + U_0 \\ &= -\frac{(Z-1)e^2}{r} - \frac{e^2 R_{\mu 0}^2}{4r^3}(3\cos^2\theta - 1) + \frac{9\varepsilon^2 m_{\mu r} e^2 R_{\mu 0}^4(1 - 2\cos^2\theta + 5\cos^4\theta)}{16 m_{er} Z r^8} \end{aligned} \tag{14}$$

Next, we consider the orbits of the electron in the plane of the orbits of the muon and the nucleus, i.e., $\theta = \pi/2$. It is easy to check, by looking at the first and second derivatives of the effective potential from Eq. (14) with respect to θ, that this position corresponds to the stable equilibrium. In this case, the effective potential will be a Coulomb potential with two terms of the $1/r^n$-perturbation (from now on we will use the atomic units $e = m_e = \hbar = 1$):

$$U_{eff} = -\frac{Z-1}{r} + \frac{R_{\mu 0}^2}{4r^3} + \frac{9\varepsilon^2 m_{\mu r} R_{\mu 0}^4}{16 m_{er} Z r^8} \tag{15}$$

The calculation of the $1/r^n$-perturbation for the Kepler problem can be found, e.g., in [14] (the treatment for the cases $n = 2$ and $n = 3$ can be found also in the textbook [15]). For the Coulomb potential $-\alpha/r$ perturbed by the potential $-\beta/r^{n+1}$, the orbit undergoes a precession with the perihelion advance

$$\delta\Phi = 2m\beta \frac{\partial}{\partial L}\left(\frac{1}{L} p^{1-n} \int_0^{\pi} (1 + \varepsilon_e \cos\varphi)^{n-1} d\varphi\right) \tag{16}$$

where m is the reduced mass of the pair "particle – Coulomb center", L is the angular momentum of the particle, ε_e is the eccentricity of its orbit, and $p = L^2/(m\alpha)$, in our case the particle being the electron. The ratio of the precession frequency to the Kepler frequency is the perihelion advance scaled by 2π, so for the second term in Eq. (15) we obtain the following ratio of the precession frequency in the plane of the orbit to the Kepler frequency

$$\frac{\omega_{pip}^{(1)}}{\omega_K} = -\frac{3}{4} m_{er}^2 (Z-1) S^2 \tag{17}$$

where

$$S = \frac{R_{\mu 0}}{L^2} \tag{18}$$

The same result could be obtained by using Eq. (1.7.10) from book [16], where the potential corresponds to the gravitational potential of the oblate Earth and is mathematically equivalent to Eq. (13) (without the last term) with the following correspondence of the quantities:

$$GMm \Leftrightarrow (Z-1)e^2, I_2 R^2 \Leftrightarrow \frac{2\gamma}{e^2}, p \Leftrightarrow \frac{Z-1}{2|E|}, \gamma = \frac{e^2 R_{\mu 0}^2}{4(Z-1)} \tag{19}$$

(the same treatment was used in paper [6]). Substituting the corresponding quantities into Eq. (1.7.10) from book [16] and considering the case of $\theta = \pi/2$, we obtain the same result as in Eq. (17) (with $e = 1$ in the atomic units).

As an example, Fig. 1 shows the plot of the ratio $rat_1 = |\omega_{pip}^{(1)}/\omega_K|$ versus $R_{\mu 0}$ for $L = 3$ and $Z = 2$. It is seen that in this ranges of $R_{\mu 0}$, the precession frequency $\omega_{pip}^{(1)}$ remains sufficiently smaller than the Kepler frequency of the electron ω_K, which is the condition of the validity of the analytical result for $\omega_{pip}^{(1)}/\omega_K$ from Eq. (17).

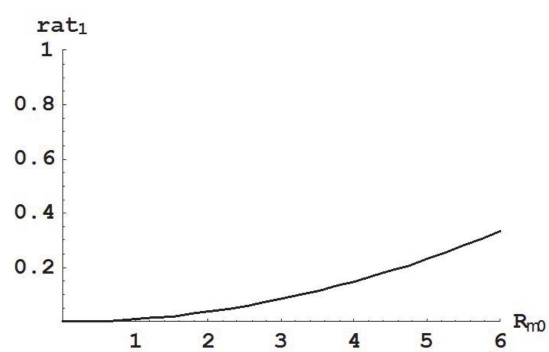

Fig. 1. Plot of the ratio $rat_1 = \omega_{pip}^{(1)}/\omega_K$ from Eq. (17) versus R_{m0} for $L = 3$ and $Z = 2$.

Applying Eq. (16) for the second perturbing term (the last term in Eq. (15)), which corresponds to the effective potential due to the low eccentricity of the muon-nucleus orbits, we obtain the following additional contribution to the precession frequency (scaled by the Kepler frequency ω_K of the electron)

$$\frac{\omega_{pip}^{(2)}}{\omega_K} = -\frac{63\varepsilon^2 m_{\mu r} m_{er}^6 (Z-1)^6 S^7 f(E_s)}{256Z}, f(E_s) = 429 - 495E_s + 135E_s^2 - 5E_s^3 \tag{20}$$

where

$$E_s = \frac{2|E|L^2}{m_{er}(Z-1)^2} \tag{21}$$

is the absolute value of the scaled dimensionless energy of the electron. For the bounded motion of the electron, the eccentricity of its orbit is $\varepsilon_e = (1 - E_s)^{1/2}$ and $0 < E_s \le 1$ (E_s is the squared ratio of the semi-minor axis to the semi-major axis of the unperturbed elliptical orbit). Concerning the function $f(E_s)$ from Eq. (20), we note that as E_s increases from 0 to 1, $f(E_s)$ monotonically decreases from 429 to 64.

As an example, Fig. 2 shows the plot of the ratio $\text{rat}_2 = |\omega_{pip}^{(2)}/\omega_K|$ versus $R_{\mu 0}$ and ε for $E_s \ll 1$ (corresponding to the relatively large eccentricity of the electron orbit $(1 - \varepsilon_e) \ll 1$), $L = 3$, and $Z = 2$ (so that $m_{\mu r} = 200.4$). It is seen that in these ranges of $R_{\mu 0}$ and ε, the additional precession frequency $\omega_{pip}^{(2)}$ remains sufficiently smaller than the Kepler frequency of the electron ω_K, which is the condition of the validity of the analytical result for $\omega_{pip}^{(2)}/\omega_K$ from Eq. (20).

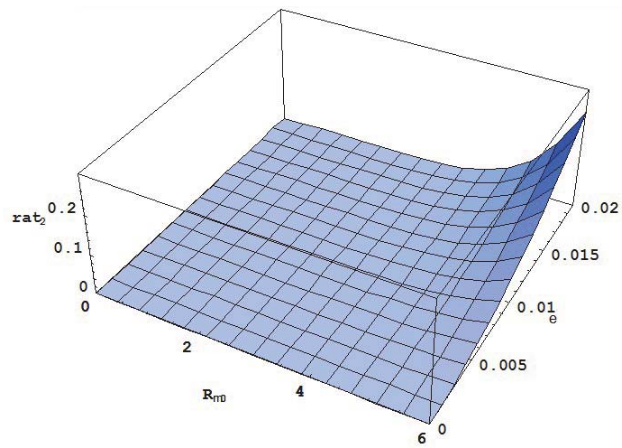

Fig. 2. Plot of the ratio $\text{rat}_2 = \omega_{pip}^{(2)}/\omega_K$ from Eq. (20) versus R_{m0} and ε for $E_s \ll 1$ (corresponding to the relatively large eccentricity of the electron orbit $(1 - e_e) \ll 1$), $L = 3$, and $Z = 2$ (so that $m_{\mu r} = 200.4$).

The ratio $\omega_{pip}^{(2)}/\omega_{pip}^{(1)}$ (denoted below as K_{21}) of the additional contribution to the precession frequency from Eq. (20) to the primary contribution to the precession frequency from Eq. (17) is:

$$K_{21} = \frac{21\varepsilon^2 m_{\mu r} m_{er}^4 (Z-1)^5 S^5 f(E_s)}{64Z} \tag{22}$$

Figure 3 shows the plot of K_{21} versus $S = R_{\mu 0}/L^2$ and E_s (defined in Eq. (21)) for $Z = 2$ and $\varepsilon = 0.02$.

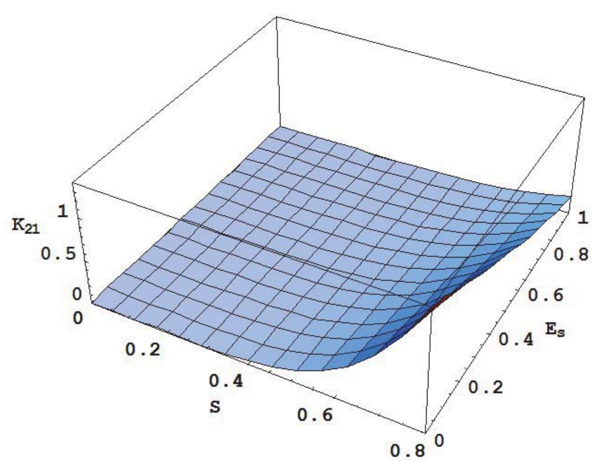

Fig. 3. Plot of the ratio K_{21} of the additional contribution to the precession frequency from Eq. (19) to the primary contribution to the precession frequency from Eq. (17) versus $S = R_{\mu 0}/L^2$ and E_s (defined in Eq. (21)) for $Z = 2$ and $\varepsilon = 0.02$.

Figure 4 presents the plot of the ratio K_{21} versus S and Z for ε = 0.02 and E_s << 1.

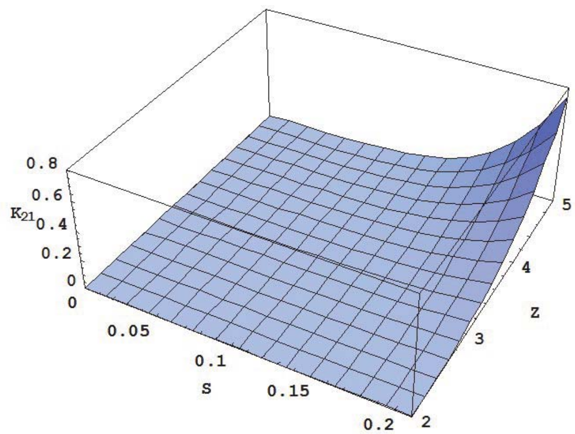

Fig. 4. Plot of the ratio K_{21} versus S and Z for ε = 0.02 and E_s << 1 (E_s << 1 corresponds to a relatively large eccentricity of the unperturbed electron orbit).

Figure 5 shows the plot of the ratio K_{21} versus $R_{\mu 0}$ and ε for Z = 2, L = 3, and E_s << 1.

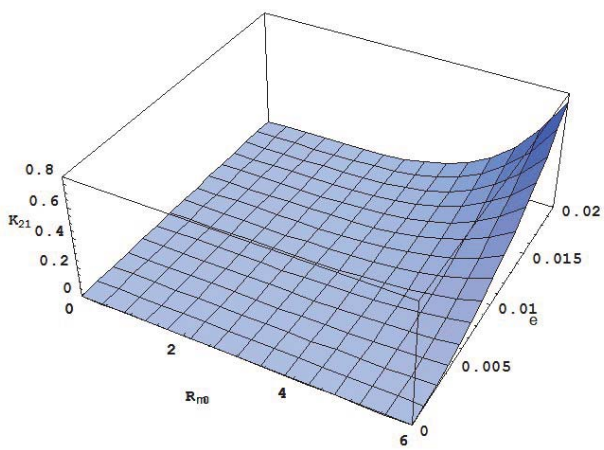

Fig. 5. Plot of the ratio K_{21} versus R_{m0} and e for Z = 2, L = 3, and E_s << 1.

From Figs. 3 – 5 it is seen that within the ranges of the parameters, where the analytical results for $\omega_{pip}^{(1)}/\omega_K$ from Eq. (17) and for $\omega_{pip}^{(2)}/\omega_K$ from Eq. (20) remain valid, the additional contribution $\omega_{pip}^{(2)}$ to the precession frequency, caused by a relatively small eccentricity of the muon orbit (and of the nucleus orbit), can reach the same order of magnitude as the primary contribution $\omega_{pip}^{(1)}$.

3. CONCLUSIONS

We considered a situation where the muon and nucleus orbits in the "nucleus-muon-electron" system are elliptical – the situation more general compared to paper [6], where the muon and nucleus orbits were set to be circular. For the case where the eccentricity ε of the muon and nucleus orbits is relatively small, we obtained an additional, ε-dependent term in the effective potential for the motion of the electron. By analytical calculations we demonstrated that in the particular case of the planar geometry (where the electron orbit is in the plane of the muon and nucleus orbits), it leads to an additional contribution to the frequency of the precession of the electron orbit. We showed that this additional, ε-dependent contribution to the precession frequency of the electron orbit can reach the same order of magnitude as the primary, ε-independent contribution to the precession frequency. Thus, the results of our paper seem to be important not only qualitatively, but also quantitatively.

References

1. C. Chelkowsky, A.D. Bandrauk, and P.B. Corkum, Laser Physics **14** (2004) 473.
2. K. Nagamine and L.I. Ponomarev, Nucl. Phys. A **721** (2003) C863.
3. J. Guffin, G. Nixon, D. Javorsek II, S. Colafrancesco, and E. Fischbach, Phys. Rev. D **66** (2002) 123508.
4. K. Nagamine, Hyperfine Interactions **138** (2001) 5.
5. L.I. Ponomarev, Contemp. Phys. **31** (1990) 219.
6. E. Oks, Can. J. Phys. **98** (2020) 857.
7. N. Kryukov and E. Oks, Canad. J. Phys. **92** (2014) 1405.
8. N. Kryukov and E. Oks, Canad. J. Phys. **91** (2013) 715.
9. B. B. Nadezhdin and E. Oks, Sov. Tech. Phys. Lett. **12** (1986) 512.
10. P. L. Kapitza, Sov. Phys. JETP **21** (1951) 588.
11. P. L. Kapitza, Uspekhi Fiz. Nauk **44** (1951) 7.
12. B. B. Nadezhdin, *Radiatsionnye i Relativistskie Effekty v Atomakh i Ionakh (Radiative and Relativistic Effects in Atoms and Ions)*, Scientific Council of the USSR Academy of Sciences on Spectroscopy, Moscow, 1986, p. 222 [in Russian].
13. E. Oks, *Analytical Advances in Quantum and Celestial Mechanics: Separating Rapid and Slow Subsystems* (IOP Publishing, Bristol, UK) 2020.
14. D. Belitz, *Lectures on Theoretical mechanics*, 2014, p. 95 https://pages.uoregon.edu/dbelitz/teaching/lecture_notes/Mechanics%20(DB)/611_2_Lecture_Notes.pdf
15. L. D. Landau and E. M. Lifshitz, *Mechanics*, Pergamon, Oxford, 1960, Chapter 3, problem 3.
16. V. V. Beletsky. *Essays on the motion of celestial bodies*. Sect. 1.7. Birkhäuser/Springer, Basel, Switzerland. 2001.