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Heavy Ions Can Have the Flavor Symmetry

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ABSTRACT: The Second Flavor of Hydrogen Atoms (SFHA) is based on the second analytical solution of the standard Dirac equation. Despite this solution is singular at small r, it can be matched with the regular solution inside the proton for the S-states, and thus becomes legitimate. The SFHA have only the S-states: so, in accordance to the selection rules of quantum mechanics they do not couple to the electromagnetic radiation (except for the 21 cm spectral line): they remain dark.

The existence of the Second Flavor of Hydrogen Atoms (SFHA), predicted theoretically in 2001, has been confirmed by four different types of atomic or molecular experiments, and evidenced by two different types of astrophysical observations. In the present paper we extended the validity of the second analytical solution of the Dirac equation outside the nucleus to heavy ions. We predicted the existence of the second flavor of heavy ions and thus the flavor symmetry of these ions. These ions have only the S-states, so that they remain dark just as the SFHA. This theoretical result has the fundamental importance in its own right. Also, it could encourage experimentalists to perform experiments for the verification of the existence of the second flavor of heavy ions. The most probable candidates are ions whose nuclei are double-magic and thus spherical. As an application of the above fundamental result, we referred to the comparison between the nuclear charge radius $r_{n,i}$, determined experimentally by the muonic x-ray transition energies, and the nuclear charge radius $r_{n,e}$, determined experimentally from the elastic electron scattering. The comparison was possible for two spherical nuclei: for $^{40}Ca_{20}$, as well as for $^{48}Ca_{20}$. In each case, $r_{n,e}$ turned out to be by about 1% smaller, than $r_{n,i}$, the difference being well beyond the experimental error margin. We showed in both above examples, that a relatively small admixture (~10%) of the second flavor of heavy ions in the target reconciles the values of the nuclear charge radius, measured by two different methods, in favor of the larger value. Finally, we noted that while the SFHA is one of the leading candidates for dark matter or a part of it, there are still unsuccessful attempts to find Weakly Interactive Massive Particles (WIMP) as a part of dark matter. We proposed that if there are massive dark matter particles, then more realistically they could be the second flavor of heavy ions, described in the present paper: they are really dark.

Key words: Dirac equation; second flavor of hydrogen atoms; second flavor of heavy ions; nuclear charge radius; dark matter

1. INTRODUCTION

One solution of the standard Dirac equation for hydrogenic atoms/ions is weakly singular at small r, while the other solution is strongly singular at small r. More specifically, the radial part $R_{Nk}(r)$ of the coordinate wave functions scales at small r as follows (see, e.g., the textbook [1]):

$$R_{Nk}(r) \sim 1/r^q, \qquad q = 1 \pm (k^2 - Z^2\alpha^2)^{1/2}. \tag{1}$$

In Eq. (1), Z is the nuclear charge, α is the fine structure constant, N is the radial quantum number, and k is the eigenvalue of the following operator commuting with the Hamiltonian:

$$K = \beta(2\mathbf{Ls} + 1), \tag{2}$$

where β is the Dirac matrix of the rank 4. In Eq. (2), **L** and **s** are the operators of the orbital momentum and spin, respectively; **Ls** is their scalar product (also known as the dot-product).

For the states of k = –1, which are the states of $l = 0$ (i.e., the S-states), Eq. (1) becomes

$$R_{N,-1}(r) \sim 1/r^q, \qquad q = 1 \pm (1 - Z^2\alpha^2)^{1/2}. \tag{3}$$

So, indeed, for $Z^2\alpha^2 << 1$, the solution, corresponding to $q = 1 - (1 - Z^2\alpha^2)^{1/2} \approx Z^2\alpha^2/2 \approx 0.000027Z^2$, is very weakly singular (hereafter, "regular"), while the second solution, corresponding to $q = 1 + (1 - Z^2\alpha^2)^{1/2} \approx 2 - Z^2\alpha^2/2 \approx 2$ is strongly singular. The second solution was usually rejected because the normalization integral diverges at r = 0.

Even after allowing for the finite nuclear size, but for the models where the assumed charge distribution inside the nucleus a uniformly charged sphere or a spherical shell, the singular solution in the exterior cannot be matched at the nucleus boundary with the regular solution in the interior and was therefore rejected as well.

In paper [2] there was derived a general class of potentials inside a spherical nucleus, such that the singular exterior solution can be actually matched at the boundary with the corresponding regular interior solution. One of the particular cases within this general class are potentials corresponding to the charge distributions within the nucleus that either have a peak at r = 0 or a plateau (or nearly a plateau) starting at r = 0, and then fall off toward the periphery.

In paper [2] the focus was on hydrogen atoms (for reasons specified below). From the well-known experiments on the elastic scattering of electrons on protons (see, e.g., [3] and [4]), it was established that the charge distribution inside protons does have a maximum at the origin and then falls off toward the periphery. As a result, it was found in paper [2] that the corresponding regular solution inside the proton can be matched with the singular solution outside the proton (i.e., with the second solution of the Dirac equation), but only for the states of $l = 0$: for the S-states. Strictly speaking, in paper [2] this was found only for the ground state, while in paper [5] this was extended to all $n^2S_{1/2}$ states, where n = N + |k| = N + 1 is the principal quantum number (n = 1, 2, 3, …), as well as to all $l = 0$ stayes of the continuous spectrum.

This alternative type of hydrogen, having only the S-states was later named the Second Flavor of Hydrogen Atoms (SFHA). Here is why. Both the wave function resulting from the regular solution and the wave function resulting from the singular solution correspond to *the same energy*. Thus, there is an *additional degeneracy*. Consequently, there should be *an additional conserved quantity* – according to the well-known fundamental theorem of quantum mechanics. To put it another way: there are *two flavors* of hydrogen atoms that differ by the eigenvalue of this additional, new conserved quantity: hydrogen atoms are characterized by the *flavor symmetry* [6], called so by analogy with the flavor symmetry of quarks.

The most significant feature of the SFHA is that due to having only the S-states, then – in accordance to the selection rules of quantum mechanics – all diagonal and non-diagonal matrix elements of the electric dipole moment operator are zeros, so that the SFHA do not couple to the electromagnetic radiation: the SFHA stay *dark* (except for the radiative transition between the two hyperfine substates of the ground state, corresponding to the 21 cm spectral line).

Here is why paper [2] dealt with hydrogen atoms. There was an enormous discrepancy between the experimental and theoretical High-energy Tail of the Linear Momentum Distribution (HTLMD) in the ground state of hydrogen atoms. Namely, the experimental HTLMD fell off much more slowly than the theoretical HTLMD: the discrepancy was almost *four orders of magnitude*. In paper [2] it was shown that alter allowing for the SFHA, this multi-order discrepancy was completely eliminated. The reason was that the much steeper rise of the SFHA coordinate wave function toward the proton at small r (compared to the usual hydrogen atoms) corresponds – according to the properties of the Fourier transform – to a *much slower fall-off* of the wave function in the linear momentum representation for large linear momenta. Since an alternative explanation was never provided, this constituted the *first experimental evidence* of the existence of the SFHA.

By now there are three additional experimental proofs of the existence of the SFHA: the proofs from of atomic or molecular experiments of three different kinds. The second proof is from the experiments on the electron impact excitation of hydrogen molecules. The most advanced calculations (by the convergent close-coupling method employing 491 states) of the excitation cross-sections for the first two stable excited electronic triplet states of H_2 – the state

$c^3\Pi_u$ and the state $a^3\Sigma_g^+$ [7] – underestimated the experimental cross-sections ([8, 9] by *a factor of two or more*. In paper [10] it was demonstrated that this very big discrepancy could be eliminated if in the experimental gas there was an admixture of about 30% of “unusual” hydrogen molecules where one or both atoms were the SFHA. The reason is that for these “unusual” H_2 molecules the corresponding calculated cross-section is by a factor of three larger than for the usual H_2 molecules. Again, since an alternative explanation was never provided, this constituted the *second experimental evidence* of the existence of the SFHA.

The third proof is from the experiments on the electron impact excitation of hydrogen atoms (rather than molecules). The theoretical ratio of the cross-sections σ_{2S}/σ_{2P} (corresponding to the excitation of the states 2S and 2P) [11] was by about 20% greater than the corresponding experimental ratio [12], while the experimental error margin was 9%. In paper [13] it was emphasized that the cross-section σ_{2S} was measured by applying an electric field intermixing the states 2P and 2S, and the registering the radiative transition from the state 2P to the state 1S (the Lyman-alpha line). However, while both the usual hydrogen atoms and the SFHA are excited to the state 2s, the intermixing of the states 2P and 2S (and the subsequent registration of the Lyman-alpha signal) happens only for the usual hydrogen lines: the SFHA, due to having only the S-states, do not contribute to the registered Lyman-alpha signal. Consequently, the cross-section σ_{2S} was underestimated compared to its actual value – in distinction to the cross-section σ_{2P} that was not influenced by the presence of the SFHA. In paper [13] it was shown that this discrepancy could be eliminated if in the experimental gas there was an admixture of about 40% of the SFHA. Again, since an alternative explanation was never provided, this constituted the *third experimental evidence* of the existence of the SFHA.

The fourth proof is from experiments dealing with the charge exchange between hydrogen atoms and low energy protons. The experimental cross-sections [14] were found to be noticeably larger than the corresponding theoretical cross-sections from paper [15]. The resonant charge exchange cross-section depends on the ionization potential U_{ioniz} from the particular atomic state (it is approximately proportional to $1/U_{ioniz}^2$). In paper [16] it was pointed out that the electric field of the incident proton increases U_{ioniz} for the usual hydrogen atoms due to the Stark shift, it does not affect the energy levels of the SFHA because for the latter all matrix elements of the electric dipole moment operator are zeros: this explained the above discrepancy. Again, since an alternative explanation was never provided, this constituted the *fourth experimental evidence* of the existence of the SFHA.

In addition to the above proofs of the existence of the SFHA from four different types of the atomic or molecular experiments, their existence is also evidenced by *two different types of astrophysical observations*. The first one was the baffling observation that the absorption in the redshifted 21 cm line from the early Universe turned out to be about *twice stronger* than anticipated from the standard cosmology [17]. This meant significantly smaller temperature of the hydrogen gas than expected from the standard cosmology. In paper [18] some unspecified dark matter was proposed as the cooling agent that chilled the hydrogen gas by collisions, and it was estimated that for the quantitative explanation of the observations [17] these dark matter particles should have the mass of the same order as baryons. In paper [5] there was analyzed the scenario where these unspecified dark matter particles were the SFHA (the latter should also contribute to the 21 cm line). It was pointed out that as the Universe expanded, the SFHA, because of having only the S-states, underwent the earlier decoupling from the cosmic microwave background radiation compared to the usual hydrogen atoms. Consequently, their spin temperature was lower than for the usual hydrogen atoms (the 21 cm line absorption signal is controlled by the spin temperature). This expounded the observed twice stronger absorption not only qualitatively, but also quantitatively [5].

The second type of astrophysical observations evidenced the existence of the SFHA was the puzzling distribution of dark matter in the Universe: from observations it turned out to be less clumpy, smoother than anticipated from Einstein’s gravitation [19]. This observation triggered appeals for new physical laws. However, in paper [20] it was shown that this befuddling observation can be expounded based on the SFHA, without the recourse to new physical laws – not only qualitatively, but also quantitatively.

As a result of the SFHA-grounded explanations of the above two perplexing astrophysical observations, the SFHA became one of the prominent nominees for dark matter or for a part of it – see, for instance, review [21].

In the present paper we show the possibility of the existence of the Second Flavor of Heavy Ions (SFHI). We also provide an example of an application of the SFHI to determining the nuclear charge radii.

2. NEW RESULTS

Bethe published the Thomas-Fermi theory for large nuclei [22] as early as in 1968. The corresponding Charge Density Distribution (CDD) had a plateau near the origin and then fell off towards the periphery – as it was shown in Fig. 6 from paper [22] by the example of $^{197}Au_{79}$. In the subsequent years, lots of further calculations of the CDD were performed for various nuclei by the Thomas-Fermi method, or by the extended Thomas-Fermi method, or by other methods – see, e.g., papers [23 - 30] and references therein. These calculations resulted in the CDD having either a peak or a plateau (or nearly a plateau) near the origin and falling off to the periphery. The exception is very heavy nuclei of the nuclear charge $Z > 80$, where the CDD exhibits a lateral maximum (see, e.g., the CDD for $^{209}Bi_{83}$ in Fig. 4 from paper [23] and then for significantly larger Z the CDD becomes a "bubble" – as illustrated, e.g., in Fig. 7 from paper [23].

So, one could state that for the range $10 < Z < 80$ (10 being conditionally chosen as the lower limit of validity of statistical approaches), the CDD in the nuclei has either a peak or a plateau (or nearly a plateau) near the origin and falls off to the periphery. This kind of CDD yields potentials satisfying the condition (derived in paper [2], under which the singular solution of the Dirac equation outside the nucleus can be matched with the regular solution inside the nucleus for the S-states of the corresponding hydrogenlike ion.

This constitutes the theoretical prediction of *possible second flavor of relatively heavy ions*. In other words, relatively heavy ions may have *flavor symmetry* – just as hydrogen atoms really have.

This theoretical result has the fundamental importance in its own right. Also, it could encourage experimentalists to perform experiments (e.g., analogous but not limited to those that proved the existence of the second flavor of hydrogen atoms) for the verification of the existence of the SFHI. Below we provide an example dealing with the influence of the SFHI on the experimental determination of the *nuclear charge radius*.

Let us start by recalling the puzzle concerning the proton charge radius r_p, the puzzle continuing for 13 years by now – since year 2010 when the formfactor experiment by the group from Mainz [31] yielded $r_p \approx 0.88$ fm in distinction to the experiment based on the muonic hydrogen spectroscopy that resulted in $r_p \approx 0.84$ fm [32]. This difference was inexplicable because it amounted to about 7 standard deviations.

In the subsequent years, various types of other experiments were conducted in this respect, such as, e.g., [33-39]. (Additional references on this issue can be looked up in review articles [40-46]. Nevertheless, the ambiguity has not been eliminated, this fact being underscored, e.g., in reviews [33, 41, 43].

In paper [47] we analyzed the situation where in addition to the muonic hydrogen spectroscopy experiments there would be also experiments on the elastic scattering of muons on hydrogen atoms. We pointed out that if there would be a difference in the proton charge radius, determined by the two different types of experiments, then the two results could be reconciled if there was a relatively small admixture of the second flavor of muonic hydrogen atoms in the experimental gas. We also provided a numerical example: even 10% admixture could result (and explain) about 4% difference in the experimentally determined parameters.

So, in the present paper we provide a similar example illustrating the possible influence of the SFHI on the experimental determination of the *nuclear charge radius* for various nucleons. Namely, we consider a mixture of the SFHI in the ratio ε to the usual heavy hydrogenlike ions of the same mass and of the same nuclear charge. In this situation, outside the nucleus, the radial part of the Dirac bispinor for the ground state can be written as follows (based on Eq. (17) from paper [2]):

$$f(r) \approx -2b^{5/4}\{1/r^{b/2} - \varepsilon[R^2/(5br^{2-b2})]\}/(1+\varepsilon^2)^{1/2}, \quad (1)$$

$$g(r) \approx 4b^{3/4}\{1/r^{b/2} - \varepsilon[R^2/(10br^{1-b/2})]\}/(1+\varepsilon^2)^{1/2}, \quad (2)$$

where

$$b = Z^2\alpha^2, \tag{3}$$

α being the fine structure constant. In Eqs. (1) and (2), R is the nucleon "sphere" radius, that is, the boundary between the regular solution of the Dirac equation in the interior region and the singular solution of the Dirac equation in the exterior region. The proportionality relation between R and the nucleon charge radius r_n is specified later on. (In Eq. (2) we corrected a misprint in the expression for g(r) from the corresponding Eq. (2) from paper [47] – in addition to entering the normalizing factor $(1+\varepsilon^2)^{1/2}$, as the denominator.)

After going through the same steps as in paper [47], the ground state energy shift ÄE due to the nucleus finite size can be expressed as follows:

$$\Delta E(\varepsilon, R) = \text{const } R^2[1/R^b - \varepsilon R/(5b) + \varepsilon^2 R^2/(100b^{2+b})]/(1 + \varepsilon^2), \tag{4}$$

where the radius R and the energy are measured in atomic units ($\hbar = m_e = e = 1$). (Here we corrected some misprints compared to paper [47].)

The next step concerns the relationship between the relative shift $\delta E = \Delta E/E$ of the ground state energy and the relative change $\delta\sigma = \Delta\sigma/\sigma$ of the elastic scattering cross-section (for the scattering of electrons on the relatively heavy hydrogenlike ions). For the elastic cross section at the limit of relatively low values of the momentum transfer – the limit relevant to the determination of the nuclear charge radius – we utilize the relation

$$\sigma = \text{const } (\langle r^2\rangle)^2, \tag{5}$$

corresponding to Eq. (115.4) from the textbook [48]. Here r is the distance of the bound electron from the nucleus; the symbol <…> means "averaged". Expression (5) is appropriate for our simple model designed just to get the message across.

The relation between ($\langle r^2\rangle$ and the unperturbed binding energy E_{bind} of the electron (i.e., for $\varepsilon = 0$) in any state (including the ground state) of hydrogenlike ions is

$$E_{bind} = \text{const}/(\langle r^2\rangle), \tag{6}$$

so that

$$(\langle r^2\rangle) = \text{const}/E_{bind}. \tag{7}$$

Therefore,

$$|\delta(\langle r^2\rangle)| = |\delta E_{bind}|, \tag{8}$$

where

$$\delta(\langle r^2\rangle) = \Delta(\langle r^2\rangle)/(\langle r^2\rangle), \qquad \delta E_{bind} = \Delta E_{bind}/E_{bind}. \tag{9}$$

From Eq. (5) it follows that

$$\delta\sigma = \Delta\sigma/\sigma = 2\ \delta(\langle r^2\rangle), \tag{10}$$

where σ is the corresponding cross-section at $\varepsilon = 0$. Then from Eqs. (10) and (8) we obtain:

$$|\delta\sigma(\varepsilon, R)| = 2|\delta E_{bind}(\varepsilon, R)|. \tag{11}$$

Because the unperturbed binding energy E_{bind} and the unperturbed cross-section σ do not depend on ε, then the change of the cross-section Δσ depends on ε in the same way as ΔE from Eq. (4) (apart from a constant)

$$\Delta\sigma(\varepsilon, R) = \text{const } R^2[1/R_p^{\ b} - \varepsilon R/(5b) + \varepsilon^2 R^{2+b}/(100b^2)]/(1 + \varepsilon^2). \tag{12}$$

If for the same nucleus there is a difference between the nuclear charge radius, determined experimentally by the muonic x-ray transition energies, and the nuclear charge radius, determined experimentally from the elastic electron scattering, such as the ratio of the latter to the former is (1 – a), where a << 1 (what could be the real situation, as specified below), then we seek the value of ε satisfying the following equation:

$$\Delta\sigma(\varepsilon, Z, R) = \Delta\sigma[0, Z, (1 - a)R]. \tag{13}$$

To put it another way, the goal of solving Eq. (12) with respect to ε is to demonstrate that from the same experimental cross-section, one can deduce either the smaller value of R while disregarding a possible share of the SFHI in the experimental target (i.e., at ε = 0) or a higher value of R at some finite value of ε.

Equation (13) is quadratic with respect to ε. It has the following solutions:

$$\varepsilon_1 = \{[1.408\text{x}10^7R^2/Z^4 + 4a(2-a)(1-2a-a^2-3.52\text{x}10^6R^2/Z^4)]^{1/2} - 3.75\text{x}10^3R/Z^2\}/[2(1-2a-a^2-3.52\text{x}10^6R^2/Z^4], \tag{14}$$

$$\varepsilon_2 = \{-[1.408\text{x}10^7R^2/Z^4 + 4a(2-a)(1-2a-a^2-3.52\text{x}10^6R^2/Z^4)]^{1/2} - 3.75\text{x}10^3R/Z^2\}/[2(1-2a-a^2-3.52\text{x}10^6R^2/Z^4]. \tag{15}$$

Only the value ε_1 is physically meaningful: the value ε_2 is negative and thus physically meaningless.

For connecting the above illustrative example with reality, we refer to the actual comparison between the nuclear charge radius, determined experimentally by the muonic x-ray transition energies, and the nuclear charge radius, determined experimentally from the elastic electron scattering. The theory of the SFHI assumes the spherical shape of the nuclei, so that it is fully applicable only to the doubly-magic (and thus spherical) nuclei, which in the above range 10 < Z < 80 are the following seven nuclei: $^{40}Ca_{20}$, $^{48}Ca_{20}$, $^{48}Ni_{28}$, $^{56}Ni_{28}$, $^{78}Ni_{28}$, $^{100}Sn_{50}$, and $^{132}Sn_{50}$. To the best of our knowledge, the nuclear charge radius, determined experimentally by the two above different methods, is available only for $^{40}Ca_{20}$ and $^{48}Ca_{20}$.

For $^{40}Ca_{20}$, the nuclear charge radius, determined experimentally by the muonic x-ray transition energies, is $r_{n,\mu}$ = 3.4813 fm (see Table IV from paper [49]), while the experimental determination from the elastic electron scattering yielded $r_{n,e}$ = 3.450±0.010 fm (see Table I from paper [50] where the latter value was deduced from the experiment by the *model independent* analysis using the Fourier-Bessel expansion for the charge distribution.) The difference $r_{n,\mu} - r_{n,e}$ = 0.031 fm, corresponding to 0.89%, is well beyond the experimental error margin. For numerically estimating the share of the SFHI in the target, we use for the nuclear charge radius r_n the value of 3.47 fm (which is the mean value between the two different measured values). After the translation into the atomic units, we get r_n = 0.0000656.

The nuclear "sphere" radius R and the nuclear charge radius r_n are proportional to each other. The value of R would be by the factor of $(5/3)^{1/2}$ greater than r_n (it would be equal to 0.0000847) if the nucleus would be a uniformly charged sphere (what the nucleus is not). The actual value of R should be between 0.0000656 and 0.0000847. By adopting the value R = 0.000075 for numerical estimates, we obtain from Eq. (13) the following:

$$\varepsilon_1 \approx 0.13. \tag{16}$$

For $^{48}Ca_{20}$ the results are very similar. Namely, one has $r_{n,\mu}$ = 3.482 [49] fm and $r_{n,e}$ = 3.451±0.009 fm [50], the latter value being also was deduced from the experiment by the *model independent* analysis using the Fourier-Bessel expansion for the charge distribution. The difference $r_{n,\mu} - r_{n,e}$ = 0.031 fm, corresponding to 0.89%, is well beyond the experimental error margin. For numerically estimating the share of the SFHI in the target, we use for the nuclear charge radius r_n the value of 3.47 fm (which is the mean value between the two different measured values). After the translation into the atomic units, we get r_n = 0.0000656.

The value of R would be by the factor of $(5/3)^{1/2}$ greater than r_n (it would be equal to 0.0000847) if the nucleus would be a uniformly charged sphere (what the nucleus is not). The actual value of R should be between 0.0000656 and 0.0000847. By adopting the value R = 0.000075 for numerical estimates, we obtain from Eq. (13) the following:

$$\varepsilon_1 \approx 0.13. \tag{17}$$

Thus, in both above examples, a relatively small admixture (~10%) of the SFHI in the target reconciles the values of the nuclear charge radius, measured by two different methods, *in favor of the larger value*.

3. CONCLUSIONS

Within the standard quantum mechanics, based on papers [2, 5], we extended the validity of the second analytical solution of the Dirac equation outside the nucleus to heavy ions. To put it another way, we predicted the existence of the second flavor of heavy ions and thus the flavor symmetry of these ions. These ions have only the S-states. Therefore, due to the selection rules of quantum mechanics they do not couple to the electromagnetic radiation: they remain dark.

This theoretical result has the fundamental importance in its own right. Also, it could encourage experimentalists to perform experiments (e.g., analogous but not limited to those that proved the existence of the second flavor of hydrogen atoms) for the verification of the existence of the second flavor of heavy ions. The most probable candidates are ions whose nuclei are double-magic and thus spherical.

As an application of the above fundamental result, we referred to the comparison between the nuclear charge radius, determined experimentally by the muonic x-ray transition energies, and the nuclear charge radius, determined experimentally from the elastic electron scattering. The comparison was possible for two spherical nuclei: for $^{40}Ca_{20}$, as well as for $^{48}Ca_{20}$. In each case, the nuclear charge radius, determined experimentally from the elastic electron scattering, turned out to be by about 1% smaller, than the nuclear charge radius, determined experimentally by the muonic x-ray transition energies, the difference being well beyond the experimental error margin. We showed in both above examples, that a relatively small admixture (~10%) of the second flavor of heavy ions in the target reconciles the values of the nuclear charge radius, measured by two different methods, in favor of the larger value.

By the way, concerning non-spherical nuclei having the ellipsoidal shape, we note in passing that in paper [51] it was shown analytically that the shift of spectral lines of muonic hydrogenlike ions can change by several times due to the ellipsoidal nuclear shape – compared to the corresponding shift for spherical nuclei. This result provided an additional method of deducing from the experiments the quadrupole moment of nuclei, as well as the standard beta-parameter related to the quadrupole moment.

The last but not least. Among two dozen theories of dark matter, only three of them do not go beyond the Standard Model and do not invent new physical laws [21]. Therefore, these three theories are favored by the Occam razor principle. The theory based on the experimentally confirmed existence of the second flavor of hydrogen atoms is one of these three and it explains more observed manifestations of dark matter than any other theory. However, none of the two dozens of theories explained by itself each and every observed manifestation of dark matter.

It is possible that dark matter is a multi-faceted phenomenon having a variety of observed manifestations – analogously, e.g., to electrons manifesting as particles in some experiments or as waves in other experiments. Therefore, theories beyond the above three could be also entertained. One of the dark matter hypotheses going beyond the Standard Model is Weakly Interactive Massive Particles (WIMP) that are assumed to be much more massive than baryons – see, e.g., reviews [52, 53]. However, despite various experimental attempts, WIMP were never observed.

We could propose that if there are massive dark matter particles, then more realistically they could be the second flavor of heavy ions, described in the present paper. Such Non-Interactive Massive Particles (NIMP) could have a doubly-magic spherical nucleus, e.g., $^{40}Ca_{20}$, and/or $^{48}Ca_{20}$, and/or $^{48}Ni_{28}$, and/or $^{56}Ni_{28}$, and/or $^{78}Ni_{28}$, and/or $^{100}Sn_{50}$, and/or $^{132}Sn_{50}$. Such NIMP are really dark, as explained in the present paper.

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